ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, a + b + c = 3$ *and* $\lambda \geq 0$ then :

$$\sum_{cyc} \frac{b^2 + c^2}{\lambda + a^2} \geq \frac{6}{\lambda + 1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{For } all\ x > 0, \lambda \geq 0, \frac{1}{x+\lambda} - \frac{1}{\lambda+1} + \frac{x-1}{(\lambda+1)^2} = \frac{1-x}{(\lambda+1)(x+\lambda)} - \frac{1-x}{(\lambda+1)^2}$$

$$= \frac{1-x}{\lambda+1}\cdot\left(\frac{1}{x+\lambda} - \frac{1}{\lambda+1}\right) = \frac{(1-x)^2}{(\lambda+1)^2(x+\lambda)} \geq 0 \therefore \frac{1}{x+\lambda} \geq \frac{1}{\lambda+1} - \frac{x-1}{(\lambda+1)^2}\ and$$

$$\text{with } x \equiv a^2, \frac{1}{a^2+\lambda} \geq \frac{1}{\lambda+1} - \frac{a^2-1}{(\lambda+1)^2}$$

$$\Rightarrow \frac{b^2+c^2}{\lambda+a^2} \geq \frac{(b^2+c^2)(\lambda+1)}{(\lambda+1)^2} - \frac{(a^2-1)(b^2+c^2)}{(\lambda+1)^2}\ and\ analogs$$

$$\Rightarrow \sum_{cyc} \frac{b^2+c^2}{\lambda+a^2} \geq \sum_{cyc} \frac{(b^2+c^2)(\lambda+1) - (a^2-1)(b^2+c^2)}{(\lambda+1)^2} \overset{?}{\geq} \frac{6(\lambda+1)}{(\lambda+1)^2}$$

$$\Leftrightarrow \lambda\left(\sum_{cyc} a^2 - 3\right) + 2\sum_{cyc} a^2 - \sum_{cyc} a^2b^2 - 3 \underset{(*)}{\overset{?}{\geq}} 0\ and, now, 2\sum_{cyc} a^2 - \sum_{cyc} a^2b^2 - 3$$

$$= \frac{2}{9}\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a\right)^2 - \sum_{cyc} a^2b^2 - \frac{3}{81}\left(\sum_{cyc} a\right)^4 \left(\because \sum_{cyc} a = 3\right)$$

$$= \frac{1}{27}\left(5\sum_{cyc} a^4 + 8\sum_{cyc} a^3b + 8\sum_{cyc} ab^3 - 21\sum_{cyc} a^2b^2\right) \geq 0; \text{since}$$

$$5\sum_{cyc} a^4 \geq 5\sum_{cyc} a^2b^2\ \text{and}\ 8\sum_{cyc} a^3b + 8\sum_{cyc} ab^3 \overset{AM-GM}{\geq} 16\sum_{cyc} a^2b^2$$

$$\therefore 2\sum_{cyc} a^2 - \sum_{cyc} a^2b^2 - 3 \overset{①}{\geq} 0\ and \sum_{cyc} a^2 - 3 \geq \frac{1}{3}\left(\sum_{cyc} a\right)^2 - 3 = \frac{9}{3} - 3 = 0$$

$$and \because \lambda \geq 0 \therefore \lambda\left(\sum_{cyc} a^2 - 3\right) \overset{②}{\geq} 0\ and\ so, ① + ② \Rightarrow (*)\ \text{is true}$$

$$\therefore \sum_{cyc} \frac{b^2+c^2}{\lambda+a^2} \geq \frac{6}{\lambda+1}\ \forall a, b, c > 0 \,|\, a + b + c = 3\ and\ \lambda \geq 0,$$

$$'' ='' \text{ iff } a = b = c = 1\ (QED)$$