ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, a + b + c = 3$ and $\lambda \geq 2$ then :

$$\sum_{cyc} \frac{(\lambda+1)a+1}{(\lambda a+1)^2} \geq \frac{3(\lambda+2)}{(\lambda+1)^2}$$

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$$\frac{(\lambda+1)a+1}{(\lambda a+1)^2} - \left(\frac{\lambda+2}{(\lambda+1)^2} + \frac{(1-a)(\lambda^2+2\lambda-1)}{(\lambda+1)^3}\right)$$

$$= \frac{-a^2\lambda^3 + a\lambda^3 - 2a^2\lambda^2 + a\lambda^2 - a\lambda + \lambda^2 + a + \lambda - 1}{(\lambda a+1)^2(\lambda+1)^2} - \frac{(1-a)(\lambda^2+2\lambda-1)}{(\lambda+1)^3}$$

$$= \frac{(1-a)(\lambda^3 a + 2\lambda^2 a + \lambda^2 + \lambda - 1)}{(\lambda a+1)^2(\lambda+1)^2} - \frac{(1-a)(\lambda^2+2\lambda-1)}{(\lambda+1)^3}$$

$$= \frac{1-a}{(\lambda+1)^2} \cdot \frac{-a^2\lambda^4 + a\lambda^4 - 2a^2\lambda^3 + a\lambda^3 + \lambda^3 + a^2\lambda^2 - 2a\lambda^2 + \lambda^2 + 2a\lambda - 2\lambda}{(\lambda a+1)^2(\lambda+1)}$$

$$= \frac{(1-a)^2}{(\lambda+1)^2} \cdot \frac{a\lambda^4 + 2a\lambda^3 - a\lambda^2 + \lambda(\lambda^2+\lambda-2)}{(\lambda a+1)^2(\lambda+1)}$$

$$= \frac{(1-a)^2\left(a\lambda^2(\lambda^2+2\lambda-1) + \lambda(\lambda-1)(\lambda+2)\right)}{(\lambda+1)^3(\lambda a+1)^2} \geq 0 \quad \forall\, a > 0 \text{ and } \forall\, \lambda \geq 1$$

$$\therefore \frac{(\lambda+1)a+1}{(\lambda a+1)^2} \geq \frac{\lambda+2}{(\lambda+1)^2} + \frac{(1-a)(\lambda^2+2\lambda-1)}{(\lambda+1)^3} \text{ and analogs}$$

$$\forall\, a, b, c > 0 \text{ and } \forall\, \lambda \geq 1 \therefore \sum_{cyc} \frac{(\lambda+1)a+1}{(\lambda a+1)^2} \geq$$

$$\frac{3(\lambda+2)}{(\lambda+1)^2} + \frac{\lambda^2+2\lambda-1}{(\lambda+1)^3} \cdot \left(3 - \sum_{cyc} a\right) \overset{a+b+c=3}{=} \frac{3(\lambda+2)}{(\lambda+1)^2} \text{ and so,}$$

$$\sum_{cyc} \frac{(\lambda+1)a+1}{(\lambda a+1)^2} \geq \frac{3(\lambda+2)}{(\lambda+1)^2} \;\forall\, a, b, c > 0 \,\middle|\, a + b + c = 3 \text{ and } \lambda \geq 1,$$

$$" = " \text{ iff } a = b = c = 1 \text{ (QED)}$$