If $a, b, c > 0, a + b + c = 3$ and $n \in \mathbb{N}$ then :

$$\sum_{cyc} \frac{a(b^n + c^n)}{\sqrt{b^{2n-1} + c^{2n-1}}} \leq 3\sqrt{2}$$

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$$\text{If } n = 0, \sum_{cyc} \frac{a(b^n + c^n)}{\sqrt{b^{2n-1} + c^{2n-1}}} = \sum_{cyc} \frac{2a.\sqrt{bc}}{\sqrt{b+c}} \overset{AM-GM}{\leq} \sum_{cyc} \frac{a(b+c)}{\sqrt{b+c}}$$

$$= \sum_{cyc} (\sqrt{ab + ac}.\sqrt{a}) \overset{CBS}{\leq} \sqrt{2\sum_{cyc} ab} . \sqrt{\sum_{cyc} a} \leq \sqrt{\frac{2}{3}\left(\sum_{cyc} a\right)^2} . \sqrt{\sum_{cyc} a} \overset{a+b+c=3}{=} \sqrt{18}$$

$$= 3\sqrt{2} \text{ and } \forall \, n \in \mathbb{N}^*, \sum_{cyc} \frac{a(b^n + c^n)}{\sqrt{b^{2n-1} + c^{2n-1}}} = \sum_{cyc} \frac{a(b^n + c^n)}{\sqrt{\frac{b^{2n}}{b} + \frac{c^{2n}}{c}}} \overset{Bergstrom}{\leq} \sum_{cyc} \frac{a(b^n + c^n)}{\sqrt{\frac{(b^n+c^n)^2}{b+c}}}$$

$$= \sum_{cyc} (\sqrt{ab + ac}.\sqrt{a}) \overset{CBS}{\leq} \sqrt{2\sum_{cyc} ab} . \sqrt{\sum_{cyc} a} \leq \sqrt{\frac{2}{3}\left(\sum_{cyc} a\right)^2} . \sqrt{\sum_{cyc} a} \overset{a+b+c=3}{=} \sqrt{18}$$

$$= 3\sqrt{2} \text{ and so, } \sum_{cyc} \frac{a(b^n + c^n)}{\sqrt{b^{2n-1} + c^{2n-1}}} \leq 3\sqrt{2} \; \forall \, a, b, c > 0 \, | \, a + b + c = 3 \text{ and } n \in \mathbb{N},$$

$$'' = '' \text{ iff } a = b = c = 1 \text{ (QED)}$$