ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0, a + b = 2$ and $\lambda \geq 2$ then :

$$\frac{1}{1-a+\lambda a^2}+\frac{1}{1-b+\lambda b^2}\geq\frac{2\sqrt{ab}}{\lambda}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{1-a+\lambda a^2}+\frac{1}{1-b+\lambda b^2}-\frac{2\sqrt{ab}}{\lambda}\overset{AM-GM}{\geq}$$

$$\frac{\lambda((a+b)^2-2ab)+2-(a+b)}{1-b+\lambda b^2-a+ab-\lambda ab^2+\lambda a^2-\lambda a^2b+\lambda^2a^2b^2}-\frac{a+b}{\lambda}$$

$$=\frac{\lambda(4-2ab)}{-1+\lambda(4-2ab)+ab-2\lambda ab+\lambda^2a^2b^2}-\frac{2}{\lambda}\ (\because a+b=2)$$

$$=\frac{\lambda^2(4-2ab)+2-2\lambda(4-2ab)-2ab+4\lambda ab-2\lambda^2a^2b^2}{\lambda(1-a+\lambda a^2)(1-b+\lambda b^2)}$$

$$=\frac{2}{\lambda(1-a+\lambda a^2)(1-b+\lambda b^2)}.\left(\lambda^2(1-ab)(2+ab)-4\lambda(1-ab)+(1-ab)\right)$$

$$=\frac{2(1-ab)}{\lambda(1-a+\lambda a^2)(1-b+\lambda b^2)}.(2\lambda^2+\lambda^2ab-4\lambda+1)$$

$$=\frac{2(1-ab)(2\lambda(\lambda-2)+\lambda^2ab+1)}{\lambda(1-a+\lambda a^2)(1-b+\lambda b^2)}\geq 0$$

$$\left(\begin{array}{c}\because 2=a+b\overset{AM-GM}{\geq}2.\sqrt{ab}\Rightarrow 1-ab\geq 0 \text{ and } \lambda-2\geq 0 \text{ and}\\ (1-a+\lambda a^2),(1-b+\lambda b^2)>0 \text{ as } >\Delta=1-4\lambda\overset{\lambda\geq 2}{\leq}-7<0\end{array}\right)$$

$$\therefore \frac{1}{1-a+\lambda a^2}+\frac{1}{1-b+\lambda b^2}\geq\frac{2\sqrt{ab}}{\lambda}\ \forall\, a,b>0\,\big|\,a+b=2 \text{ and } \lambda\geq 2,$$

$$''=''\ \text{iff } a=b=1\ (\text{QED})$$