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If $a, b, c > 0, a+b+c = 3$ and $\lambda \ge 0$ then :

$$\sum_{cyc} \frac{a^2}{bc^2} + \frac{\lambda}{abc} \ge \frac{4(\lambda+3)abc}{ab+bc+ca+abc}$$

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$$3 = \sum_{cyc} a \overset{AM-GM}{\ge} 3.\sqrt[3]{abc} \Rightarrow abc \le 1 \text{ and } \sum_{cyc} ab \overset{AM-GM}{\ge} 3.\sqrt[3]{a^2b^2c^2}$$

$$\overset{abc \le 1}{\ge} 3abc \therefore \sum_{cyc} ab \overset{①}{\ge} 3abc;\ \text{now, } \sum_{cyc} \frac{a^2}{bc^2} - \frac{12abc}{\sum_{cyc} ab + abc}$$

$$= \sum_{cyc} \frac{\left(\frac{a}{c}\right)^2}{b} - \frac{12}{\frac{\sum_{cyc} ab}{abc} + 1} \overset{\substack{\text{Bergstrom} \\ \text{and} \\ \text{via } ①}}{\ge} \frac{\left(\sum_{cyc} \frac{b}{a}\right)^2}{\sum_{cyc} a} - \frac{12}{3+1} \overset{\substack{AM-GM \\ \text{and} \\ \because a+b+c=3}}{\ge} \frac{9}{3} - 3 = 0$$

$$\therefore \sum_{cyc} \frac{a^2}{bc^2} \overset{②}{\ge} \frac{12abc}{\sum_{cyc} ab + abc};\ also\ \frac{\lambda}{abc} - \frac{4\lambda abc}{\sum_{cyc} ab + abc} \overset{abc \le 1}{\ge} \lambda - \frac{4\lambda}{\frac{\sum_{cyc} ab}{abc} + 1} \overset{\text{via } ①}{\ge}$$

$$\lambda - \frac{4\lambda}{3+1}\ (\because \lambda \ge 0) = 0 \therefore \frac{\lambda}{abc} \overset{③}{\ge} \frac{4\lambda abc}{\sum_{cyc} ab + abc} \text{ and so, } ② + ③ \Rightarrow$$

$$\sum_{cyc} \frac{a^2}{bc^2} + \frac{\lambda}{abc} \ge \frac{4(\lambda+3)abc}{ab+bc+ca+abc}\ \forall a, b, c > 0 \mid a+b+c = 3 \text{ and } \lambda \ge 0,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$