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If $a, b, c > 0, a+b+c=3$ and $\lambda \geq 1$ then :

$$\sum_{cyc} \frac{a^2 - a + \lambda}{b+1} \geq \frac{3\lambda}{2}$$

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$$\sum_{cyc}\left((a^2-a)(a+1)(c+1)\right) \overset{?}{\geq} 0 \Leftrightarrow \sum_{cyc} ab^3 + \sum_{cyc} a^3 - \sum_{cyc} ab - \sum_{cyc} a \underset{(*)}{\overset{?}{\geq}} 0$$

and since $\sum_{cyc} ab^3 = abc\sum_{cyc}\frac{b^2}{c} \overset{\text{Bergstrom}}{\geq} \frac{abc\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a} = abc\sum_{cyc} a$ $\therefore$ in order to prove $(*)$, it suffices to prove : $abc\sum_{cyc} a + \sum_{cyc} a^3 - \sum_{cyc} ab - \sum_{cyc} a \overset{?}{\geq} 0$

$$\Leftrightarrow abc\sum_{cyc} a + \frac{1}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^3\right) - \frac{1}{9}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^2 - \frac{1}{27}\left(\sum_{cyc} a\right)^4 \overset{?}{\geq} 0$$

$$\left(\because \sum_{cyc} a = 3\right) \Leftrightarrow 4\sum_{cyc} a^4 + \sum_{cyc} a^3 b + \sum_{cyc} ab^3 \overset{?}{\geq} 6\sum_{cyc} a^2b^2 \rightarrow \text{true}$$

$$\because 4\sum_{cyc} a^4 \geq 4\sum_{cyc} a^2b^2 \text{ and } \sum_{cyc} a^3 b + \sum_{cyc} ab^3 \overset{\text{AM-GM}}{\geq} 2\sum_{cyc} a^2b^2 \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\left((a^2-a)(a+1)(c+1)\right) \geq 0 \Rightarrow \sum_{cyc}\frac{a^2-a}{b+1} \geq 0 \Rightarrow \sum_{cyc}\frac{a^2-a+\lambda}{b+1} \geq$$

$$\lambda.\sum_{cyc}\frac{1}{b+1} \overset{\text{Bergstrom}}{\geq} \frac{9\lambda}{\sum_{cyc} a + 3}\ (\because \lambda \geq 1 > 0) \overset{a+b+c=3}{=} \frac{9\lambda}{3+3} \text{ and so,}$$

$$\sum_{cyc}\frac{a^2-a+\lambda}{b+1} \geq \frac{3\lambda}{2}\ \forall\, a,b,c > 0 \,\middle|\, a+b+c = 3 \text{ and } \lambda \geq 1,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$