ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } a,b,c > \frac{1}{\lambda+1}, a+b+c \le 3 \text{ and } \lambda \ge 0 \text{ then :}$$

$$\sum_{cyc} \frac{1}{\log_2(\lambda a+b)} \ge \frac{3}{\log_2(\lambda+1)}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\lambda a + b > \frac{\lambda}{\lambda+1} + \frac{1}{\lambda+1} = 1 \therefore \ln(\lambda a+b) > 0 \text{ and analogs}$$

$$\therefore \sum_{cyc} \frac{1}{\log_2(\lambda a+b)} = (\ln 2).\sum_{cyc} \frac{1}{\ln(\lambda a+b)} \overset{\text{Bergstrom}}{\ge} \frac{9(\ln 2)}{\sum_{cyc} \ln(\lambda a+b)} \overset{?}{\ge} \frac{3}{\log_2(\lambda+1)}$$

$$= \frac{3(\ln 2)}{\ln(\lambda+1)} \Leftrightarrow 3\ln(\lambda+1) \overset{?}{\ge} \sum_{cyc} \ln(\lambda a+b) \ (\because \ln 2 > 0) \Leftrightarrow \sum_{cyc} \ln\left(\frac{\lambda a+b}{\lambda+1}\right) \underset{(*)}{\overset{?}{\le}} 0$$

$$\left(\because \frac{\lambda a+b}{\lambda+1} > 0 \text{ as } \lambda a+b \ge 1 > 0 \text{ and } \lambda \ge 0 \Rightarrow \lambda+1 \ge 1 > 0\right)$$

$$\text{Now, } \ln\left(\frac{\lambda a+b}{\lambda+1}\right) \le \frac{\lambda a+b}{\lambda+1} - 1 \text{ and analogs} \therefore \sum_{cyc} \ln\left(\frac{\lambda a+b}{\lambda+1}\right) \le$$

$$\frac{\lambda}{\lambda+1}.\sum_{cyc} a + \frac{1}{\lambda+1}.\sum_{cyc} a - 3 = \sum_{cyc} a - 3 \overset{a+b+c\le 3}{\le} 3 - 3 = 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{1}{\log_2(\lambda a+b)} \ge \frac{3}{\log_2(\lambda+1)} \ \forall\, a,b,c > \frac{1}{\lambda+1} \,\Big|\, a+b+c \le 3 \text{ and } \lambda \ge 0,$$

$$'' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$