ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x + y + z = 3$ and $\lambda \ge 3$ then :

$$\sum_{cyc} \frac{\lambda - x^3}{x} \ge 3(\lambda - 1)$$

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$$\sum_{cyc} \frac{\lambda - x^3}{x} \overset{?}{\ge} 3(\lambda - 1) \Leftrightarrow \lambda\left(\sum_{cyc} \frac{1}{x} - 3\right) \overset{?}{\ge} \sum_{cyc} x^2 - 3$$

$$\overset{x+y+z=3}{\Leftrightarrow} \lambda\left(\frac{1}{3}\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 3xyz\right) \overset{?}{\ge} xyz\left(\frac{9\sum_{cyc} x^2}{\left(\sum_{cyc} x\right)^2} - 3\right)$$

$$\Leftrightarrow \frac{\lambda}{3}\cdot\left(\sum_{cyc} x\right)^2\cdot\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz\right) \overset{?}{\ge} 3xyz\left(3\sum_{cyc} x^2 - \left(\sum_{cyc} x\right)^2\right) \text{ and}$$

$$\because \frac{\lambda}{3} \ge 1 \wedge \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz \overset{\text{AM-GM}}{\ge} 0 \therefore \text{it suffices to prove :}$$

$$\left(\sum_{cyc} x\right)^2\cdot\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz\right) \boxed{\underset{(*)}{\overset{?}{\ge}}} 3xyz\left(3\sum_{cyc} x^2 - \left(\sum_{cyc} x\right)^2\right)$$

Assigning $y + z = a, z + x = b, x + y = c \Rightarrow a + b - c = 2z > 0, b + c - a = 2x > 0$ and $c + a - b = 2y > 0 \Rightarrow a + b > c, b + c > a, c + a > b \Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\Rightarrow \sum_{cyc} x = s, xyz = r^2 s, \sum_{cyc} xy = 4Rr + r^2 \text{ and } \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \Rightarrow (*) \Leftrightarrow$$

$$s^2(s(4Rr + r^2) - 9r^2 s) \overset{?}{\ge} 3r^2 s(3(s^2 - 8Rr - 2r^2) - s^2)$$

$$\Leftrightarrow (2R - 7r)s^2 + 9r^2(4R + r) \boxed{\underset{(**)}{\overset{?}{\ge}}} 0 \text{ and it's trivially true when : } 2R - 7r \ge 0$$

and when : $2R - 7r < 0$, then : LHS of $(**) \overset{\text{Gerretsen}}{\ge}$
$(2R - 7r)(4R^2 + 4Rr + 3r^2) + 9r^2(4R + r) = 2(R - 2r)(4R^2 - 2Rr + 3r^2) \ge 0$

$$\because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{\lambda - x^3}{x} \ge 3(\lambda - 1)$$

$$\forall\, x, y, z > 0 \,\middle|\, x + y + z = 3 \wedge \lambda \ge 3, '' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$