ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x + y + z = 3$ *and* $\lambda \ge \frac{1}{2}$ then :

$$\lambda \sum_{cyc} x^3 \ge \sum_{cyc} x^2 + \sum_{cyc} x + 3(\lambda - 2)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\lambda \sum_{cyc} x^3 \overset{?}{\ge} \sum_{cyc} x^2 + \sum_{cyc} x + 3(\lambda - 2) \overset{x+y+z=3}{\Leftrightarrow}$$

$$\lambda\left(\sum_{cyc} x^3 - \frac{1}{9}\left(\sum_{cyc} x\right)^3\right) \overset{?}{\ge} \frac{1}{3}\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2\right) + \frac{1}{9}\left(\sum_{cyc} x\right)^3 - \frac{6}{27}\left(\sum_{cyc} x\right)^3$$

$$\Leftrightarrow \lambda\left(9\sum_{cyc} x^3 - \left(\sum_{cyc} x\right)^3\right) \overset{?}{\ge} 3\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2\right) - \left(\sum_{cyc} x\right)^3 \text{ and}$$

$$\because \lambda \ge \frac{1}{2} \wedge 9\sum_{cyc} x^3 - \left(\sum_{cyc} x\right)^3 \overset{\text{Holder}}{\ge} 0 \therefore \text{it suffices to prove :}$$

$$\frac{1}{2}\left(9\sum_{cyc} x^3 - \left(\sum_{cyc} x\right)^3\right) \overset{?}{\ge} 3\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2\right) - \left(\sum_{cyc} x\right)^3$$

$$\Leftrightarrow 9\sum_{cyc} x^3 + \left(\sum_{cyc} x\right)^3 \overset{?}{\ge} 6\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2\right)$$

$$\Leftrightarrow 4\sum_{cyc} x^3 + 6xyz \underbrace{\overset{?}{\ge}}_{(*)} 3\sum_{cyc} x^2y + 3\sum_{cyc} xy^2$$

$$\text{Now, } 3\sum_{cyc} x^3 + 9xyz \underbrace{\overset{\text{Schur}}{\ge}}_{①} 3\sum_{cyc} x^2y + 3\sum_{cyc} xy^2 \text{ and } \sum_{cyc} x^3 \underbrace{\overset{\text{AM-GM}}{\ge}}_{②} 3xyz \text{ and}$$

$$① + ② \Rightarrow (*) \text{ is true} \therefore \lambda \sum_{cyc} x^3 \ge \sum_{cyc} x^2 + \sum_{cyc} x + 3(\lambda - 2)$$

$$\forall x, y, z > 0 \mid x + y + z = 3 \text{ and } \lambda \ge \frac{1}{2}, '' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$