ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z, p, q, r \in R,\ p+q+r=1\ and\ \ x^p y^q z^r = 1$ then:

$$\frac{p^2x^2}{qy+zr}+\frac{q^2y^2}{px+zr}+\frac{r^2z^2}{px+qy}\geq\frac{1}{2}$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

Lemma 1: **Cauchy-Schwarz Inequality**

$$\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)\geq\left(\sum_{i=1}^{n}a_ib_i\right)^2,\qquad \begin{matrix}a_1,a_2,\dots,a_n\in R\\ b_1,b_2,\dots,b_n\in R\end{matrix}$$

Lemma 2: **Weighted AM-GM Inequality (General Form)**

For any positive real numbers $x_1, x_2, \dots, x_n$ and positive weights $w_1, w_2, \dots, w_n$

such that $\sum_{i=1}^{n}w_i=1$ **the following inequality holds:**

$$\sum_{i=1}^{n}w_ix_i\geq\prod_{i=1}^{n}x_i^{w_i}$$

$$a_1=\frac{px}{\sqrt{qy+zr}},a_2=\frac{qy}{\sqrt{px+zr}},a_3=\frac{rz}{\sqrt{px+qy}}$$
$$b_1=\sqrt{qy+zr},b_2=\sqrt{px+zr},b_3=\sqrt{px+qy}$$

$$(a_1^2+a_2^2+b_2^2)(b_1^2+b_2^2+b_3^2)\overset{LEMMA\ 1}{\geqq}(px+qy+zr)^2$$

$$\underbrace{\left(\frac{p^2x^2}{qy+zr}+\frac{q^2y^2}{px+zr}+\frac{z^2r^2}{px+qy}\right)}_{LHS}(2px+2qx+2zr)\geq(px+qy+zr)^2$$

$$LHS\geq\frac{px+qy+zr}{2}$$

$$\begin{matrix}w_1=p,w_2=q,w_3=r\\ x_1=x,x_2=y,x_3=z\end{matrix}\rightarrow\sum_{i=1}^{3}w_i=1,\qquad\sum_{i=1}^{3}w_ix_i\geq\prod_{i=1}^{3}x_i^{w_i}$$

$$px+qy+rz\overset{LEMMA\ 2}{\geqq}x^py^qz^r=1$$

$$LHS \geq \frac{px+qy+rz}{2} \geq \frac{1}{2} \rightarrow \frac{p^2x^2}{qy+zr} + \frac{q^2y^2}{px+zr} + \frac{r^2z^2}{px+qy} \geq \frac{1}{2} \quad (Q.E.D)$$

$$\text{Equality holds for } \begin{cases} p=q=r=\frac{1}{3} \\ x=y=z=1 \end{cases}$$