ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c,d>0$ then:

$$\sum_{cyc}\frac{1}{a^3+b^3}\ge\frac{192}{(a+b+c+d)^3}$$

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$$\sum_{i=1}^{n}\frac{a_i^2}{x_i}\overset{BERGSTROM}{\ge}\frac{\left(\sum_{i=1}^{n}a_i\right)^2}{\sum_{i=1}^{n}x_i}\quad a_i=1\rightarrow\sum_{i=1}^{n}\frac{1}{x_i}\ge\frac{n^2}{\sum_{i=1}^{n}x_i}$$

$$n=6\rightarrow\sum_{i=1}^{6}\frac{1}{x_i}\ge\frac{36}{\sum_{i=1}^{6}x_i}\quad\sum_{cyc}\frac{1}{a^3+b^3}\ge\frac{36}{\sum_{cyc}(a^3+b^3)}$$

$$\sum_{cyc}\frac{1}{a^3+b^3}\ge\frac{36}{3(a^3+b^3+c^3+d^3)}=\frac{12}{a^3+b^3+c^3+d^3}$$

The Power Mean Inequality (Generalized Mean Inequality)

$$M_p=\left(\frac{1}{n}\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}},\quad a_1,a_2\dots a_n\in R^+\quad p>q\quad M_p\ge M_q$$

$$M_3\ge M_1\rightarrow n=4\qquad\sqrt[3]{\frac{a^3+b^3+c^3+d^3}{4}}\ge\frac{a+b+c+d}{4}$$

$$\frac{a^3+b^3+c^3+d^3}{4}\ge\frac{(a+b+c+d)^3}{64}\rightarrow\sum_{cyc}a^3\ge\frac{(a+b+c+d)^3}{16}$$

$$\sum_{cyc}\frac{1}{a^3+b^3}\ge\frac{12}{a^3+b^3+c^3+d^3}\quad\sum_{cyc}\frac{1}{a^3+b^3}\ge\frac{12}{\frac{(a+b+c+d)^3}{16}}=\frac{192}{(a+b+c+d)^3}$$

$$a=b=c=d\rightarrow\frac{6}{2a^3}\ge\frac{192}{(4a)^3}\quad\frac{3}{a^3}\ge\frac{3}{a^3}$$

$$\sum_{cyc}\frac{1}{a^3+b^3}\ge\frac{192}{(a+b+c+d)^3}$$