

# ROMANIAN MATHEMATICAL MAGAZINE

*If  $a, b, c > 0$  then prove that:*

$$\frac{a+b+c}{\sqrt[3]{abc}} \geq 2 + \sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$

*Proposed by Le Anh Tu-Vietnam*

*Solution by Tapas Das-India*

$$\text{Let } x = \frac{a^2+b^2+c^2}{\sqrt[2]{a^2b^2c^2}} \stackrel{AM-GM}{\geq} \frac{3\sqrt[3]{a^2b^2c^2}}{\sqrt[3]{a^2b^2c^2}} = 3$$

$$y = \frac{ab+bc+ca}{\sqrt[3]{a^2b^2c^2}} \stackrel{AM-GM}{\geq} \frac{3\sqrt[3]{a^2b^2c^2}}{\sqrt[3]{a^2b^2c^2}} = 3. \text{ and } z = \frac{a^2+b^2+c^2}{ab+bc+ca} \geq 1 \text{ clearly, } \frac{x}{y} = z$$

*We need to show:*

$$\frac{a+b+c}{\sqrt[3]{abc}} \geq 2 + \sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}} \text{ or } \sqrt{\frac{(a+b+c)^2}{\sqrt[3]{a^2b^2c^2}}} \geq 2 + \sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$

$$\sqrt{\frac{a^2+b^2+c^2+2(ab+bc+ca)}{\sqrt[3]{a^2b^2c^2}}} \geq 2 + \sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$

$$\sqrt{\frac{a^2+b^2+c^2}{\sqrt[2]{a^2b^2c^2}}} + 2 \frac{ab+bc+ca}{\sqrt[3]{a^2b^2c^2}} \geq 2 + \sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$$

$$\sqrt{x+2y} \geq 2 + \sqrt{z} \text{ or } \sqrt{x+2y} \geq 2 + \sqrt{\frac{x}{y}} \text{ or } \sqrt{y\left(\frac{x}{y}+2\right)} \geq 2 + \sqrt{\frac{x}{y}}$$

$$\sqrt{3\left(\frac{x}{y}+2\right)} \stackrel{y \geq 3}{\geq} 2 + \sqrt{\frac{x}{y}} \text{ or } 3\left(\frac{x}{y}+2\right) \geq 4 + \frac{x}{y} + 4\sqrt{\frac{x}{y}} \text{ or } 2\left(\frac{x}{y}-2\sqrt{\frac{x}{y}}+1\right) \geq 0$$

$$\text{or } \left(\sqrt{\frac{x}{y}}-1\right)^2 \geq 0 \text{ True. Equality holds for } a=b=c$$