

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\prod_{cyc} (\sqrt{a^2 + 3b^2} - a) \leq \frac{2abc\sqrt[3]{abc}}{a + b + c - \sqrt[3]{abc}}$$

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$$\begin{aligned} \text{Let } x = \sqrt[3]{abc} \text{ \& } A = \frac{a}{x}, B = \frac{b}{x}, C = \frac{c}{x} \text{ then } ABC = 1 \\ s = a + b + c = Ax + Bx + Cx = x(A + B + C) = xP, \\ (P = A + B + C), Q = AB + BC + CA \\ Q^2 \geq 3ABC(A + B + C) = 3P \text{ \& } P \geq 3\sqrt[3]{ABC} = 3 \bullet \\ a^2 + 3b^2 = \frac{a^2}{1} + \frac{b^2}{1} + \frac{b^2}{1} + \frac{b^2}{1} \stackrel{\text{Radon}}{\geq} \frac{(a + b + b + b)^2}{4} = \frac{(a + 3b)^2}{4} \quad (1) \\ \sqrt{a^2 + 3b^2} - a = \frac{(\sqrt{a^2 + 3b^2} - a)(\sqrt{a^2 + 3b^2} + a)}{\sqrt{a^2 + 3b^2} + a} = \\ = \frac{3b^2}{\sqrt{a^2 + 3b^2} + a} \stackrel{(1)}{\leq} \frac{3b^2}{\frac{a+3b}{2} + a} = \frac{2b^2}{a + b} \\ L.H.S \leq \frac{8a^2b^2c^2}{(a + b)(b + c)(c + a)} \quad (A), \\ \frac{\prod(a + b)}{x^3} = (A + B)(B + C)(C + A) = \sum A \sum AB - ABC = PQ - 1 \\ \frac{4x^2(s - x)}{x^3} = 4x^2 \cdot \frac{x(P - 1)}{x^3} = 4(P - 1) \end{aligned}$$

We need to show:

$$\begin{aligned} \frac{8a^2b^2c^2}{(a + b)(b + c)(c + a)} &\leq \frac{2abc\sqrt[3]{abc}}{a + b + c - \sqrt[3]{abc}} = \frac{2abcx}{s - x} \\ \text{or, } \frac{\prod(a + b)}{4abc} &\geq \frac{s - x}{x} = \frac{4x^2(s - x)}{x^3} \text{ or, } \frac{\prod(a + b)}{x^3} \geq \frac{4x^2(s - x)}{x^3} \text{ or,} \\ PQ - 1 &\geq 4(P - 1) \\ \text{or, } P\sqrt[3]{3P} &\stackrel{Q \geq 3P}{\geq} 4P - 3 \text{ or, } 3P^3 - 16P^2 + 24P - 9 \geq 0 \text{ or,} \\ (P - 3)(3P^2 - 7P + 3) &\geq 0 \\ \text{or, } (P - 3)(P(3P - 7) + 3) &\geq 0 \text{ true as } P \geq 3 \end{aligned}$$

Equality holds for $a=b=c$