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Let a, b, c be positive real numbers. Prove the following inequalities:

$$\sum_{cyc} \sqrt{a^2 + ab + b^2} \geq \sqrt{4 \sum_{cyc} a^2 + 5 \sum_{cyc} ab}$$

$$\sum_{cyc} \sqrt{a^2 + ab + b^2} \leq \sqrt{5 \sum_{cyc} a^2 + 4 \sum_{cyc} ab}$$

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Proof of (1):

Let $L = \sum_{cyc} \sqrt{a^2 + ab + b^2}$.

$$L^2 = \sum_{cyc} (a^2 + ab + b^2) + 2 \sum_{cyc} \sqrt{a^2 + ab + b^2} \sqrt{b^2 + bc + c^2}$$

By the Cauchy-Schwarz inequality:

$$\begin{aligned} \sqrt{a^2 + ab + b^2} \sqrt{b^2 + bc + c^2} &= \sqrt{\left(b + \frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}a}{2}\right)^2} \sqrt{\left(b + \frac{c}{2}\right)^2 + \left(\frac{\sqrt{3}c}{2}\right)^2} \\ &\geq \left(b + \frac{a}{2}\right) \left(b + \frac{c}{2}\right) + \frac{3ac}{4} \\ &= b^2 + \frac{1}{2}ab + \frac{1}{2}bc + ac \end{aligned}$$

$$2 \sum_{cyc} \sqrt{a^2 + ab + b^2} \sqrt{b^2 + bc + c^2} \geq 2 \sum_{cyc} \left(b^2 + \frac{1}{2}ab + \frac{1}{2}bc + ac\right) = 2 \sum_{cyc} a^2 + 4 \sum_{cyc} ab$$

$$\begin{aligned} L^2 &\geq \left(2 \sum_{cyc} a^2 + \sum_{cyc} ab\right) + \left(2 \sum_{cyc} a^2 + 4 \sum_{cyc} ab\right) \\ &= 4 \sum_{cyc} a^2 + 5 \sum_{cyc} ab \Rightarrow (1) \end{aligned}$$

Proof of (2):

By the Cauchy-Schwarz inequality:

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$$L^2 = \left(\sum_{cyc} \sqrt{a+b} \sqrt{\frac{a^2+ab+b^2}{a+b}} \right)^2 \leq \left(\sum_{cyc} (a+b) \right) \left(\sum_{cyc} \frac{a^2+ab+b^2}{a+b} \right)$$

$$L^2 \leq 2(a+b+c) \sum_{cyc} \left(a+b - \frac{ab}{a+b} \right) = 4(a+b+c)^2 - 2(a+b+c) \sum_{cyc} \frac{ab}{a+b}$$

We need to prove:

$$4(a+b+c)^2 - 2(a+b+c) \sum_{cyc} \frac{ab}{a+b} \leq 5 \sum_{cyc} a^2 + 4 \sum_{cyc} ab$$

$$\sum_{cyc} a^2 - 4 \sum_{cyc} ab + 2(a+b+c) \sum_{cyc} \frac{ab}{a+b} \geq 0$$

$$\sum_{cyc} a^2 - 2 \sum_{cyc} ab + 2abc \sum_{cyc} \frac{1}{a+b} \geq 0$$

Let $p = a+b+c$, $q = ab+bc+ca$, $r = abc$. The inequality becomes:

$$p^2 - 4q + 2r \left(\frac{p^2+q}{pq-r} \right) \geq 0$$

$$(p^2 - 4q)(pq - r) + 2r(p^2 + q) \geq 0 \quad (\text{since } pq - r > 0)$$

$$p^3q - 4pq^2 + p^2r + 6qr \geq 0$$

Using Schur's inequality of the third degree ($p^3 - 4pq + 9r \geq 0$) and the identity $p^2 \geq 3q$:

$$p^3q - 4pq^2 + p^2r + 6qr \geq p^3q - 4pq^2 + 3qr + 6qr = q(p^3 - 4pq + 9r) \geq 0 \Rightarrow (2)$$