

ROMANIAN MATHEMATICAL MAGAZINE

Let a, b, c be positive real numbers. Prove that:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{3}{4} + \frac{(a+b+c)(ab+bc+ca)}{(a+b+c)^3 + 9abc} + \frac{\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b}}{a^2 + b^2 + c^2}$$

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$$\begin{aligned} \sum_{cyc} \frac{a(a^2 + b^2 + c^2)}{b+c} - \sum_{cyc} \frac{a^3}{b+c} &= \sum_{cyc} \frac{a(b^2 + c^2)}{b+c} = \sum_{cyc} \frac{a[(b+c)^2 - 2bc]}{b+c} = \sum_{cyc} \left(a(b+c) - \frac{2abc}{b+c} \right) = \\ &= 2(ab+bc+ca) - 2abc \sum_{cyc} \frac{1}{b+c} \stackrel{\text{Cauchy}}{\geq} \\ &\leq 2(ab+bc+ca) - \frac{9abc}{a+b+c} \left\{ as \sum_{cyc} \frac{1}{b+c} \stackrel{\text{Bergstrom}}{\geq} \frac{9}{2(a+b+c)} \right\} \\ \therefore \sum_{cyc} \frac{a}{b+c} - \frac{\sum \frac{a^3}{b+c}}{\sum a^2} &\leq \frac{2(ab+bc+ca) - \frac{9abc}{a+b+c}}{a^2 + b^2 + c^2} = \frac{2(a+b+c)(ab+bc+ca) - 9abc}{(a+b+c)(a^2 + b^2 + c^2)} \end{aligned}$$

Let $p := a + b + c$, $q = ab + bc + ca$ and $r = abc$.

Also assume $S = p^3 - 4pq + 9r$ which is $\stackrel{\text{Schur}}{\geq} 0$. $\therefore S \geq 0$

So, we need to show that: $\frac{2pq - 9r}{p(p^2 - 2q)} \leq \frac{3}{4} + \frac{(a+b+c)(ab+bc+ca)}{(a+b+c)^3 + 9abc} = \frac{3}{4} + \frac{pq}{p^3 + 9r}$

We have:

$$\begin{aligned} \frac{2pq - 9r}{p(p^2 - 2q)} &= 1 - \frac{p^3 - 4pq + 9r}{p(p^2 - 2q)} = 1 - \frac{S}{p(p^2 - 2q)} \text{ and } \frac{3}{4} + \frac{pq}{p^3 + 9r} = \\ &= 1 - \frac{p^3 - 4pq + 9r}{4(p^3 + 9r)} = 1 - \frac{S}{4(p^3 + 9r)} \end{aligned}$$

Thus, it suffices to prove that: $1 - \frac{S}{p(p^2 - 2q)} \leq 1 - \frac{S}{4(p^3 + 9r)}$

$$\Leftrightarrow S \cdot 4(p^3 + 9r) - S \cdot p(p^2 - 2q) \geq 0 \Leftrightarrow S(3p^3 + 2pq + 36r) \geq 0$$

Which is true as $S \geq 0$. Equality holds if and only if $a = b = c$.