

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\frac{x^3 + y^3 + z^3}{(x + y + z)(x^2 + y^2 + z^2)} \geq \frac{1}{3} + \frac{(x - y)^2 + (y - z)^2 + (z - x)^2}{6(x^2 + y^2 + z^2)}$$

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Schur's inequality : $a^3 + b^3 + c^3 + 3abc \geq a^2(b + c) + b^2(c + a) + c^2(a + b)$

$$a^3 + b^3 + c^3 + 3abc = (a + b + c)(a^2 + b^2 + c^2) - (a^3 + b^3 + c^3)$$

$$\text{or, } 2(a^3 + b^3 + c^3) + 3abc \geq (a + b + c)(a^2 + b^2 + c^2) \quad (A)$$

We need to show:

$$\frac{x^3 + y^3 + z^3}{(x + y + z)(x^2 + y^2 + z^2)} \geq \frac{1}{3} + \frac{(x - y)^2 + (y - z)^2 + (z - x)^2}{6(x^2 + y^2 + z^2)}$$

$$\text{or } \frac{x^3 + y^3 + z^3}{(x + y + z)(x^2 + y^2 + z^2)} - \frac{(x - y)^2 + (y - z)^2 + (z - x)^2}{6(x^2 + y^2 + z^2)} \geq \frac{1}{3}$$

$$L.H.S = \frac{x^3 + y^3 + z^3}{(x + y + z)(x^2 + y^2 + z^2)} - \frac{(x - y)^2 + (y - z)^2 + (z - x)^2}{6(x^2 + y^2 + z^2)} =$$

$$= \frac{x^3 + y^3 + z^3}{(x + y + z)(x^2 + y^2 + z^2)} - \frac{(x^2 + y^2 + z^2) - (xy + yz + zx)}{3(x^2 + y^2 + z^2)} =$$

$$= \frac{3(x^3 + y^3 + z^3) - (x + y + z)(x^2 + y^2 + z^2) + (x + y + z)(xy + yz + zx)}{3(x^2 + y^2 + z^2)(x + y + z)} =$$

$$= \frac{3(x^3 + y^3 + z^3) - (x^3 + y^3 + z^3) - x^2(y + z) - y^2(z + x) - z^2(x + y) + (x + y + z)(xy + yz + zx)}{3(x^2 + y^2 + z^2)(x + y + z)}$$

$$= \frac{2(x^3 + y^3 + z^3) - x^2(y + z) - y^2(z + x) - z^2(x + y) + 3xyz + x^2(y + z) + y^2(z + x) + z^2(x + y)}{3(x^2 + y^2 + z^2)(x + y + z)}$$

$$= \frac{2(x^3 + y^3 + z^3) + 3xyz}{3(x + y + z)(x^2 + y^2 + z^2)} \stackrel{\text{By (A)}}{\geq} \frac{1}{3}$$

Equality holds for $x=y=z$.