

ROMANIAN MATHEMATICAL MAGAZINE

If a, b, c be positive real numbers then prove that:

$$\left(\sqrt{a^2 + 8bc} - a\right)\left(\sqrt{b^2 + 8ca} - b\right)\left(\sqrt{c^2 + 8ab} - c\right) \leq \frac{64a^2b^2c^2}{(a + \sqrt[3]{abc})(b + \sqrt[3]{abc})(c + \sqrt[3]{abc})}$$

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$$\left(\sqrt{a^2 + 8bc} - a\right) = \frac{(\sqrt{a^2 + 8bc} + a)(\sqrt{a^2 + 8bc} - a)}{(\sqrt{a^2 + 8bc} + a)} = \frac{8bc}{(\sqrt{a^2 + 8bc} + a)}$$

$$\text{Similarly: } \left(\sqrt{b^2 + 8ca} - b\right) = \frac{8ca}{(\sqrt{b^2 + 8ca} + b)}$$

$$\left(\sqrt{c^2 + 8ab} - c\right) = \frac{8ab}{(\sqrt{c^2 + 8ab} + c)}$$

$$\begin{aligned} &\left(\sqrt{a^2 + 8bc} - a\right)\left(\sqrt{b^2 + 8ca} - b\right)\left(\sqrt{c^2 + 8ab} - c\right) = \\ &= \frac{512a^2b^2c^2}{(\sqrt{a^2 + 8bc} + a)(\sqrt{b^2 + 8ca} + b)(\sqrt{c^2 + 8ab} + c)} \end{aligned}$$

We need to show:

$$\begin{aligned} &\left(\sqrt{a^2 + 8bc} - a\right)\left(\sqrt{b^2 + 8ca} - b\right)\left(\sqrt{c^2 + 8ab} - c\right) \leq \\ &\leq \frac{64a^2b^2c^2}{(a + \sqrt[3]{abc})(b + \sqrt[3]{abc})(c + \sqrt[3]{abc})} \\ &= \frac{512a^2b^2c^2}{(\sqrt{a^2 + 8bc} + a)(\sqrt{b^2 + 8ca} + b)(\sqrt{c^2 + 8ab} + c)} \leq \\ &\leq \frac{64a^2b^2c^2}{(a + \sqrt[3]{abc})(b + \sqrt[3]{abc})(c + \sqrt[3]{abc})} \end{aligned}$$

$$\begin{aligned} &\left(\sqrt{a^2 + 8bc} + a\right)\left(\sqrt{b^2 + 8ca} + b\right)\left(\sqrt{c^2 + 8ab} + c\right) \geq \\ &\geq 8(a + \sqrt[3]{abc})(b + \sqrt[3]{abc})(c + \sqrt[3]{abc}) \quad (A) \end{aligned}$$

$$\text{Lemma: } \forall m > 0, \sqrt{m^2 + \frac{8}{m}} + m \geq \frac{2(m+1)}{\sqrt{m}}$$

$$\text{Proof: } \sqrt{m^2 + \frac{8}{m}} + m \geq \frac{2(m+1)}{\sqrt{m}} \text{ or } \sqrt{m^3 + 8} + m^{\frac{3}{2}} \geq 2m + 2$$

$$\text{or } \sqrt{t^6 + 8} + t^3 \stackrel{t^2=x}{\geq} 2t^2 + 2 \text{ or } \left(\sqrt{t^6 + 8}\right)^2 \geq (2t^2 + 2 - t^3)^2$$

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$$4(t^5 - t^4 + t^3 - 2t^2 + 1) \geq 0 \text{ or } 4(t-1)^2(t^3 + t^2 + 2t + 1) \geq 0 \text{ True}$$

$$\text{Let } x = \frac{a}{\sqrt[3]{abc}}, y = \frac{b}{\sqrt[3]{abc}}, z = \frac{c}{\sqrt[3]{abc}} \text{ then } xyz = 1$$

$$\begin{aligned} (\sqrt{a^2 + 8bc} + a) &= \sqrt[3]{abc} \left(\sqrt{\frac{a^2}{(abc)^{\frac{2}{3}}} + \frac{8bc}{(abc)^{\frac{2}{3}}} + \frac{a}{\sqrt[3]{abc}}} \right) = \\ &= \sqrt[3]{abc} \left(\sqrt{\frac{a^2}{(abc)^{\frac{2}{3}}} + \frac{8}{\frac{a}{\sqrt[3]{abc}}} + \frac{a}{\sqrt[3]{abc}}} \right) = \\ &= \sqrt[3]{abc} \left(\sqrt{x^2 + \frac{8}{x} + x} \right) \stackrel{\text{Lemma}}{\geq} \sqrt[3]{abc} \frac{2(x+1)}{\sqrt{x}} \end{aligned}$$

$$\begin{aligned} &(\sqrt{a^2 + 8bc} + a)(\sqrt{b^2 + 8ca} + b)(\sqrt{c^2 + 8ab} + c) \geq \\ &\geq \sqrt[3]{abc} \frac{2(x+1)}{\sqrt{x}} \sqrt[3]{abc} \frac{2(y+1)}{\sqrt{y}} \sqrt[3]{abc} \frac{2(z+1)}{\sqrt{z}} = \\ &= \frac{8(\sqrt[3]{abc})^3 (x+1)(y+1)(z+1)}{\sqrt{xyz}} = \\ &\stackrel{xyz=1}{=} 8(\sqrt[3]{abc})^3 (x+1)(y+1)(z+1) = \\ &= 8(\sqrt[3]{abc})^3 \left(\frac{a}{\sqrt[3]{abc}} + 1 \right) \left(\frac{b}{\sqrt[3]{abc}} + 1 \right) \left(\frac{c}{\sqrt[3]{abc}} + 1 \right) = \\ &= 8(a + \sqrt[3]{abc})(b + \sqrt[3]{abc})(c + \sqrt[3]{abc}), \text{ so (A) is true} \end{aligned}$$

Equality holds for $a=b=c$.