

ROMANIAN MATHEMATICAL MAGAZINE

Let a, b, c positive real numbers with $abc = 1$. Prove that:

$$(a^2 + b + c)(b^2 + c + a)(c^2 + a + b) \geq 3(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$\begin{aligned} (a + b + c) &\stackrel{AM-GM}{\geq} 3\sqrt[3]{abc} \stackrel{abc=1}{=} 3 \text{ or, } x \stackrel{a+b+c=x}{\geq} 3 \quad (1) \\ (1 + a + b)(1 + b + c)(1 + c + a) &\stackrel{AM-GM}{\leq} \left(\frac{(1 + a + b) + (1 + b + c) + (1 + c + a)}{3} \right)^3 \\ &= \left(\frac{3 + 2(a + b + c)}{3} \right)^3 \stackrel{(1)}{\leq} \left(\frac{(a + b + c) + 2(a + b + c)}{3} \right)^3 = (a + b + c)^3 \quad (2) \\ (a + b + c)^2 &\geq 3(ab + bc + ca) \quad (3) \\ (a^2 + b + c)(1 + b + c) &\stackrel{C-S}{\geq} (a + b + c)^2 \text{ or, } (a^2 + b + c) \geq \frac{(a + b + c)^2}{(1 + b + c)} \\ \text{Similarly: } (b^2 + c + a) &\geq \frac{(a + b + c)^2}{1 + c + a} \text{ \& } (c^2 + a + b) \geq \frac{(a + b + c)^2}{1 + a + b} \\ (a^2 + b + c)(b^2 + c + a)(c^2 + a + b) &\geq \frac{(a + b + c)^6}{(1 + a + b)(1 + b + c)(1 + c + a)} \geq \\ &\stackrel{(2)}{\geq} \frac{(a + b + c)^6}{(a + b + c)^3} = (a + b + c)^3 = \\ &= (a + b + c)(a + b + c)^2 \stackrel{(3)}{\geq} 3(a + b + c)(ab + bc + ca) = \\ &\stackrel{abc=1}{=} 3(a + b + c) \frac{(ab + bc + ca)}{abc} = 3(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \end{aligned}$$

Equality holds for $a=b=c=1$.