ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ then prove that:

$$\sqrt{1+\frac{8bc}{a^2}}+\sqrt{1+\frac{8ca}{b^2}}+\sqrt{1+\frac{8ab}{c^2}}\geq 3\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$(L.H.S)^2=\sum_{cyc}\left(1+\frac{8bc}{a^2}\right)+2\sum_{cyc}\sqrt{\left(1+\frac{8bc}{a^2}\right)\left(1+\frac{8ca}{b^2}\right)}\overset{C-S}{\geq}$$

$$\geq\sum_{cyc}\left(1+\frac{8bc}{a^2}\right)+2\sum_{cyc}\left(1+\frac{8c}{\sqrt{ab}}\right)=9+8\sum_{cyc}\frac{bc}{a^2}+16\sum_{cyc}\frac{c}{\sqrt{ab}}$$

$$(R.H.S)^2=9(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=27+9\sum_{sym}\left(\frac{a}{b}+\frac{b}{a}\right)$$

$$we\ need\ to\ show:\ 9+8\sum_{cyc}\frac{bc}{a^2}+16\sum_{cyc}\frac{c}{\sqrt{ab}}\geq\ 27+9\sum_{sym}\left(\frac{a}{b}+\frac{b}{a}\right)$$

$$or,8\sum_{cyc}\frac{x^2}{yz}+16\sum_{cyc}\frac{\sqrt{xy}}{z}\geq 18+9\sum_{sym}\left(\frac{x}{y}+\frac{y}{x}\right)\ \left(let\ x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}\right)$$

$$or,8\sum_{cyc}x^3+16\sum_{cyc}(xy)^{\frac{3}{2}}\geq 18xyz+9\sum_{sym}x^2y\ \ (A)$$

$$now,2(xy)^{\frac{3}{2}}+z^3\overset{AM-GM}{\geq}3\sqrt[3]{(xy)^3,z^3}=3xyz\ then\ \sum_{cyc}x^3+2\sum_{cyc}(xy)^{\frac{3}{2}}\geq 9xyz$$

$$x^3+y^3+4(xy)^{\frac{3}{2}}=\left(x^3+(xy)^{\frac{3}{2}}+(xy)^{\frac{3}{2}}\right)+\left(y^3+(xy)^{\frac{3}{2}}+(xy)^{\frac{3}{2}}\right)\overset{AM-GM}{\geq}3x^2y+3y^2x$$

$$then\ 2\sum_{cyc}x^3+4\sum_{cyc}(xy)^{\frac{3}{2}}\geq 3\sum_{sym}x^2y$$

$$now\ from\ (A)\ we\ get$$

$$:8\sum_{cyc}x^3+16\sum_{cyc}(xy)^{\frac{3}{2}}=2\left(\sum_{cyc}x^3+2\sum_{cyc}(xy)^{\frac{3}{2}}\right)+3\left(2\sum_{cyc}x^3+4\sum_{cyc}(xy)^{\frac{3}{2}}\right)\geq$$

$$\geq 2\times 9xyz+3\times 3\sum_{sym}x^2y=18xyz+9\sum_{sym}x^2y\ proof\ complete.$$

$$Equality\ holds\ for\ a=b=c.$$