

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\frac{(a+b)^2+1}{(a+b)^3+1} + \frac{(a+b)^2+1}{(a+b)^3+1} + \frac{(a+b)^2+1}{(a+b)^3+1} \leq \frac{a+b+c}{\sqrt{2abc}}$$

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Lemma: $\forall x > 0$,

$$\frac{x^2+1}{x^3+1} \leq \frac{1}{\sqrt{x}}$$

Proof:

$$(x^3+1)^2 - x(x^2+1)^2 = x^6 - x^5 - x + 1 = (x-1)(x^5-1) \geq 0$$

$$\text{so } \frac{(x^3+1)^2}{x^2+1} \geq \frac{1}{x} \text{ or } \frac{x^2+1}{x^3+1} \leq \frac{1}{\sqrt{x}}$$

$$\frac{(a+b)^2+1}{(a+b)^3+1} + \frac{(a+b)^2+1}{(a+b)^3+1} + \frac{(a+b)^2+1}{(a+b)^3+1} = \sum \frac{(a+b)^2+1}{(a+b)^3+1} \leq$$

$$\stackrel{\text{Lemma}}{\leq} \sum \frac{1}{\sqrt{a+b}} \stackrel{CBS}{\leq} \sqrt{3 \left(\sum \frac{1}{a+b} \right)} \stackrel{AM-HM}{\leq} \sqrt{\frac{3}{4} \sum \left(\frac{1}{a} + \frac{1}{b} \right)} = \sqrt{\frac{3}{2} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)} =$$

$$= \sqrt{\frac{3ab+bc+ca}{2abc}} \leq \sqrt{\frac{\frac{(a+b+c)^2}{3}}{abc}} \cdot \frac{3}{2} = \frac{a+b+c}{\sqrt{2abc}}$$

Equality holds for $a = b = c = \frac{1}{2}$.