

ROMANIAN MATHEMATICAL MAGAZINE

Let a, b, c positive real numbers with $abc = 1$. Prove that:

$$\sqrt[3]{(a^2 + b + c)(b^2 + c + a)(c^2 + a + b)} \geq \sqrt{\frac{3}{2} \left(a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)}$$

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$$\begin{aligned} (a + b + c) &\stackrel{AM-GM}{\geq} 3\sqrt[3]{abc} \stackrel{abc=1}{=} 3 \text{ or, } x^{a+b+c=x} \geq 3 \\ (a^2 + b + c)(1 + b + c) &\stackrel{C-S}{\geq} (a + b + c)^2 \text{ or, } (a^2 + b + c) \geq \frac{(a + b + c)^2}{(1 + b + c)} \\ \text{Similarly: } (b^2 + c + a) &\geq \frac{(a + b + c)^2}{1 + c + a} \text{ \& } (c^2 + a + b) \geq \frac{(a + b + c)^2}{1 + a + b} \\ \sqrt[3]{(a^2 + b + c)(b^2 + c + a)(c^2 + a + b)} &\geq \sqrt[3]{\frac{(a + b + c)^2(b + c + a)^2(c + a + b)^2}{(1 + b + c)(1 + c + a)(1 + a + b)}} \\ &= \frac{(a + b + c)^2}{\sqrt[3]{(1 + b + c)(1 + c + a)(1 + a + b)}} \stackrel{AM-GM}{\geq} \frac{\frac{(a+b+c)^2}{(1+b+c)+(1+c+a)+(1+a+b)}}{3} = \\ &= \frac{3(a + b + c)^2}{3 + 2(a + b + c)} = \frac{3x^2}{3 + 2x} \text{ so, } L.H.S \geq \frac{3x^2}{3 + 2x} \\ \sqrt{\frac{3}{2} \left(a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)} &= \sqrt{\frac{3}{2} \left(a + b + c + \frac{ab + bc + ca}{abc} \right)} \stackrel{abc=1}{=} \\ &= \sqrt{\frac{3}{2} (a + b + c + ab + bc + ca)} \leq \sqrt{\frac{3}{2} \left(a + b + c + \frac{(a + b + c)^2}{3} \right)} = \\ &= \sqrt{\frac{3}{2} \left(x + \frac{x^2}{3} \right)} = \sqrt{\frac{x}{2} (x + 3)} \text{ so, } R.H.S \leq \sqrt{\frac{x}{2} (x + 3)} \end{aligned}$$

We need to show: $\frac{3x^2}{3 + 2x} \geq \sqrt{\frac{x}{2} (x + 3)}$ or $\left(\frac{3x^2}{3 + 2x} \right)^2 \geq \left(\sqrt{\frac{x}{2} (x + 3)} \right)^2$

$$18x^3 \geq (3 + 2x)^2(3 + x) \text{ or } 14x^3 - 24x^2 - 45x - 27 \geq 0$$

$$(x - 3)(14x^2 + 18x + 9) \geq 0 \text{ True as } x \geq 3$$

Equality holds for $a=b=c=1$