

ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, xy + yz + zx + xyz = 4$ then:

$$\frac{1 - \sqrt{1 - xyz}}{1 + \sqrt{1 - xyz}} \leq \min\{x, y, z\} \leq 1 - \sqrt{1 - xyz}$$

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$$\text{Let } x = \frac{2a}{b+c}, y = \frac{2b}{c+a}, z = \frac{2c}{a+b} \text{ where } a, b, c > 0$$

WLOG $a \leq b \leq c$ then $b+c \geq c+a \geq a+b$ then

$$x \leq y \leq z \text{ \& } \min\{x, y, z\} = x = \frac{2a}{b+c}$$

as $a \leq b \leq c$ and $b+c \geq c+a \geq a+b$ then $b+c \geq 2a$ & $0 < x \leq 1$

$$\text{Let } P = xyz = \frac{8abc}{(a+b)(b+c)(c+a)} \stackrel{\text{Cesaro}}{\leq} \frac{8abc}{8abc} = 1(A)$$

We need to show:

$$\min\{x, y, z\} \leq 1 - \sqrt{1 - xyz} \text{ or, } x \leq 1 - \sqrt{1 - xyz} \text{ or, } 1 - xyz \leq (1 - x)^2$$

$$\text{or, } 1 - P \leq 1 - 2x + x^2 \text{ or, } P \geq x(2 - x) \text{ or, } xyz \geq x(2 - x) \text{ or, } yz \geq 2 - x$$

$$\text{or, } \left(\frac{2b}{c+a}\right)\left(\frac{2c}{a+b}\right) \geq 2 - \left(\frac{2a}{b+c}\right) \text{ or, } \frac{2bc}{(a+b)(a+c)} \geq \frac{b+c-a}{b+c}$$

$$\text{or, } 2bc(b+c) \geq (a+b)(a+c)(b+c-a)$$

$$2bc(b+c) \geq a(b+c)^2 + bc(b+c) - a^3 - abc$$

$$a^3 - a(b+c)^2 + abc + bc(b+c) \geq 0$$

$$a(a+b+c)(a-b-c) + bc(a+b+c) \geq 0$$

$$(a+b+c)(a-b)(a-c) \geq 0$$

True as $a \leq b \leq c$.

Equality case: holds iff $(a-b)(a-c) = 0$, since $a \leq b \leq c$ then $a = b$ which translate to $x = y \leq z$

We need to show :

$$\frac{1 - \sqrt{1 - xyz}}{1 + \sqrt{1 - xyz}} \leq \min\{x, y, z\} \text{ or } \frac{1 - \sqrt{1 - P}}{1 + \sqrt{1 - P}} \leq x$$

$$1 - x \leq \sqrt{1 - P}(1 + x) \text{ or } (1 - x)^2 \leq (1 - P)(1 + x)^2$$

$$P(1 + x)^2 \leq (1 + x)^2 - (1 - x)^2 \text{ or } P(1 + x)^2 \leq 4x \text{ or } P \leq \frac{4x}{(1 + x)^2}$$

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$$xyz \leq \frac{4x}{(1+x)^2}$$

$$yz(1+x)^2 \leq 4 \text{ or } \frac{4bc}{(c+a)(a+b)} \cdot \frac{(b+c+2a)^2}{(b+c)^2} \leq 4$$

$$\frac{bc}{(c+a)(a+b)} \frac{(b+c+2a)^2}{(b+c)^2} \leq 1 \text{ true}$$

$$\begin{aligned} \text{since } bc(b+c+2a)^2 &= bc((b+c)^2 + 4a(b+c) + 4a^2) = \\ &= bc(b+c)^2 + 4abc(b+c) + 4a^2bc \stackrel{AM-GM}{\leq} \\ &\leq bc(b+c)^2 + 4a \frac{(b+c)^2}{4} (b+c) + 4a^2 \frac{(b+c)^2}{4} = (b+c)^2(a^2 + ab + ac + bc) \end{aligned}$$

$$= (b+c)^2(a+b)(a+c) \text{ then } \frac{bc}{(c+a)(a+b)} \frac{(b+c+2a)^2}{(b+c)^2} \leq 1)$$

Equality case: hold for $b = c$ which implies that $x \leq y = z$