

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$a^2 + b^2 + c^2 \geq ab + bc + ca + 3 \sqrt[3]{\frac{(a-b)^2(b-c)^2(c-a)^2}{4}}$$

Determine all cases of equality

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Solution by Tapas Das-India

$$\begin{aligned} & \text{WLOG } a \geq b \geq c \text{ let } x = a - b \geq 0, y = b - c \geq 0 \\ & (c - a) = -(a - c) = -((a - b) + (b - c)) = -(x + y) \\ & a^2 + b^2 + c^2 - (ab + bc + ca) = \frac{1}{2}((a - b)^2 + (b - c)^2 + (c - a)^2) \\ & = \frac{1}{2}(x^2 + y^2 + (x + y)^2) = x^2 + xy + y^2 \\ & = \frac{1}{2}(x^2 + y^2) + \frac{1}{2}(x^2 + y^2) + xy \geq \\ & \stackrel{AM-GM}{\geq} \frac{1}{2}(x^2 + y^2) + \frac{1}{2} \times 2xy + xy = \frac{1}{2}(x^2 + y^2) + 2xy = \\ & = \left(\frac{1}{2}x^2 + \frac{1}{2}xy\right) + \left(\frac{1}{2}y^2 + \frac{1}{2}xy\right) + xy = \frac{1}{2}x(x + y) + \frac{1}{2}y(x + y) + xy \stackrel{AM-GM}{\geq} \\ & \stackrel{AM-GM}{\geq} 3 \sqrt[3]{\frac{1}{2}x(x + y) \cdot \frac{1}{2}y(x + y)xy} = 3 \sqrt[3]{\frac{x^2y^2(x + y)^2}{4}} = 3 \sqrt[3]{\frac{(a - b)^2(b - c)^2(c - a)^2}{4}} \\ & \text{Equality } x = y \text{ or } a - b = b - c \text{ or } 2b = a + c \\ & x = y = 0 \text{ or } a = b = c. \end{aligned}$$