

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, $a^2 + b^2 + c^2 = 2(ab + bc + ca)$ then:

$$\frac{\sqrt{bc}}{a + (\sqrt{b} - \sqrt{c})^2} + \frac{\sqrt{ca}}{b + (\sqrt{c} - \sqrt{a})^2} + \frac{\sqrt{ab}}{c + (\sqrt{a} - \sqrt{b})^2} \geq \frac{(a + b + c)^3}{24abc}$$

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$$\text{Let } x = \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}} \stackrel{AM-GM}{\geq} 2$$

WLOG a is the largest variable.

$$a^2 + b^2 + c^2 = 2(ab + bc + ca) \text{ or, } a^2 + (b - c)^2 - 2a(b + c) = 0$$

$$\text{or, } a^2 - 2a(b + c) + (b + c)^2 = (b + c)^2 - (b - c)^2 \text{ or, } (a - (b + c))^2 = 4bc$$

$$a - (b + c) = 2\sqrt{bc} \text{ or, } a = b + c + 2\sqrt{bc}$$

$$a = (\sqrt{b} + \sqrt{c})^2 \text{ or, } \sqrt{a} = \sqrt{b} + \sqrt{c} \text{ (A)}$$

$$\frac{\sqrt{bc}}{a + (\sqrt{b} - \sqrt{c})^2} \stackrel{\text{by (A)}}{=} \frac{\sqrt{bc}}{(\sqrt{b} + \sqrt{c})^2 + (\sqrt{b} - \sqrt{c})^2} = \frac{\sqrt{bc}}{2(b + c)} = \frac{1}{2} \frac{1}{\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}} = \frac{1}{2x}$$

$$\frac{\sqrt{ca}}{b + (\sqrt{c} - \sqrt{a})^2} = \frac{\sqrt{c}\sqrt{a}}{b + (\sqrt{c} - \sqrt{a})^2} \stackrel{\text{by (A)}}{=} \frac{\sqrt{c}(\sqrt{b} + \sqrt{c})}{b + b} = \frac{1}{2} \left(\frac{c}{b} + \sqrt{\frac{c}{b}} \right)$$

$$\frac{\sqrt{ab}}{c + (\sqrt{a} - \sqrt{b})^2} = \frac{\sqrt{a}\sqrt{b}}{c + (\sqrt{a} - \sqrt{b})^2} \stackrel{\text{by (A)}}{=} \frac{\sqrt{b}(\sqrt{b} + \sqrt{c})}{c + c} = \frac{1}{2} \left(\frac{b}{c} + \sqrt{\frac{b}{c}} \right)$$

$$\begin{aligned} L.H.S &= \frac{\sqrt{bc}}{a + (\sqrt{b} - \sqrt{c})^2} + \frac{\sqrt{ca}}{b + (\sqrt{c} - \sqrt{a})^2} + \frac{\sqrt{ab}}{c + (\sqrt{a} - \sqrt{b})^2} \\ &= \frac{1}{2x} + \frac{1}{2} \left(\frac{c}{b} + \sqrt{\frac{c}{b}} \right) + \frac{1}{2} \left(\frac{b}{c} + \sqrt{\frac{b}{c}} \right) = \frac{1}{2x} + \frac{1}{2} \left(\frac{c}{b} + \frac{b}{c} \right) + \frac{1}{2} \left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}} \right) = \\ &= \frac{1}{2x} + \frac{1}{2} (x^2 - 2) + \frac{x}{2} = \frac{x^3 + x^2 - 2x + 1}{2x} \end{aligned}$$

$$R.H.S = \frac{(a + b + c)^3}{24abc} = \frac{\left((\sqrt{b} + \sqrt{c})^2 + (b + c) \right)^3}{24(\sqrt{b} + \sqrt{c})^2 bc} = \frac{(2(b + c) + 2\sqrt{bc})^3}{24bc(b + c + 2\sqrt{bc})} =$$

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$$= \frac{8 (b + c + \sqrt{bc})^3}{24 bc(b + c + 2bc)} = \frac{1}{3} \frac{\left(\left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}} \right) + 1 \right)^3}{\left(\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}} \right) + 2} = \frac{1}{3} \frac{(x + 1)^3}{x + 2}$$

We need to show:

$$\frac{x^3 + x^2 - 2x + 1}{2x} \geq \frac{1}{3} \frac{(x + 1)^3}{x + 2} \text{ or, } 3(x + 2)(x^3 + x^2 - 2x + 1) \geq 2x(x + 1)^3$$

or, $x^4 + 3x^3 - 6x^2 - 11x + 6 \geq 0$ *or,* $(x - 2)(x^3 + 5x^2 + 4x - 3) \geq 0$ *True as* $x \geq 2$

Equality holds permutation of $(a, b, c) = (4t, t, t)$ *for* $t > 0$.