

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 2$ then:

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} + \sqrt{2abc} \leq a + b + c + 4abc$$

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$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 2 \text{ or, } ab + bc + ca + 2abc = 1$$

$$\text{We take } a = \frac{x}{y+z}, b = \frac{y}{z+x}, c = \frac{z}{x+y}$$

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} = \sum \sqrt{ab} = \sum \sqrt{\frac{x}{y+z} \cdot \frac{y}{z+x}} = \sum \sqrt{\frac{y}{y+z} \cdot \frac{x}{z+x}} \leq$$

$$\stackrel{AM-GM}{\leq} \frac{1}{2} \sum \left(\frac{y}{y+z} + \frac{x}{x+z} \right) = \frac{1}{2} \sum \left(\frac{x}{x+z} + \frac{z}{x+z} \right) = \frac{3}{2}$$

$$(x+y)(y+z)(z+x) \stackrel{Cesaro}{\geq} 8xyz \text{ or } \frac{1}{8} \geq \frac{x}{y+z} \cdot \frac{y}{z+x} \cdot \frac{z}{x+y},$$

$$abc \leq \frac{1}{8} \text{ or } \sqrt{2abc} \leq \frac{1}{2}$$

$$L.H.S = \sqrt{ab} + \sqrt{bc} + \sqrt{ca} + \sqrt{2abc} \leq \frac{3}{2} + \frac{1}{2} = 2$$

$$\begin{aligned} R.H.S &= a + b + c + 4abc = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \frac{4xyz}{(x+y)(y+z)(z+x)} \\ &= \frac{x(x+y)(x+z) + y(y+z)(y+x) + z(z+x)(z+y) + 4xyz}{(x+y)(y+z)(z+x)} \\ &= \frac{\sum x^3 + \sum_{sym} x^2y + 7xyz}{(x+y)(y+z)(z+x)} = \frac{\sum x^3 + \sum_{sym} x^2y + 7xyz}{\sum x^2y + 2xyz} \end{aligned}$$

We need to show:

$$\frac{\sum x^3 + \sum_{sym} x^2y + 7xyz}{\sum x^2y + 2xyz} \geq 2 \text{ or } \sum x^3 + 3xyz \geq \sum x^2y$$

True since it is Schur's inequality of degree 3

$$\text{Equality holds for } a = b = c = \frac{1}{2}$$