

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\sum_{cyc} \frac{(a-b)^2}{a^2+b^2} + 3 \left(\frac{ab+bc+ca}{a^2+b^2+c^2} \right)^2 \leq 3$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

$$\begin{aligned} \sum_{cyc} \frac{(a-b)^2}{a^2+b^2} &= 3 - 2 \sum_{cyc} \frac{ab}{a^2+b^2} = 3 - 2 \sum_{cyc} \frac{a^2b^2}{ab(a^2+b^2)} \leq \\ &\stackrel{\text{Bergstrom}}{\leq} 3 - \frac{2(ab+bc+ca)^2}{ab(a^2+b^2) + bc(b^2+c^2) + ca(a^2+c^2)} \end{aligned}$$

We need to show:

$$3 - \frac{2(ab+bc+ca)^2}{ab(a^2+b^2) + bc(b^2+c^2) + ca(a^2+c^2)} + 3 \left(\frac{ab+bc+ca}{a^2+b^2+c^2} \right)^2 \leq 3$$

$$3 - \frac{2(ab+bc+ca)^2}{(a^2+b^2+c^2)(ab+bc+ca) - abc(a+b+c)} + 3 \left(\frac{ab+bc+ca}{a^2+b^2+c^2} \right)^2 \leq 3$$

$$3(a^2+b^2+c^2)(ab+bc+ca) - 3abc(a+b+c) \leq 2(a^2+b^2+c^2)$$

$$2(a^4+b^4+c^4) + 4(a^2b^2+b^2c^2+c^2a^2) \geq 3 \sum_{cyc} a^3(b+c) + 3abc \sum a - 3abc \sum a$$

$$\sum (a^4 - 3a^3b + 4a^2b^2 - 3ab^3 + b^4) \geq 0 \text{ or,}$$

$$\sum (a-b)^2(a^2-ab+b^2) \geq 0 \text{ True}$$

Equality holds for $a=b=c$.