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Let a, b, c be positive real numbers. Prove the following inequalities:

$$\sqrt{a^2 + 3bc} + \sqrt{b^2 + 3ca} + \sqrt{c^2 + 3ab} \geq a + b + c + \sqrt{3(ab + bc + ca)}$$

$$\sqrt{(a^2 + 3bc)(b^2 + 3ca)(c^2 + 3ab)} \geq (a + \sqrt{bc})(b + \sqrt{ca})(a + \sqrt{ab})$$

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Solution to problem 1:

Let $p = \sqrt{a^2 + 3bc} + a$, $q = \sqrt{b^2 + 3ca} + b$, and $r = \sqrt{c^2 + 3ab} + c$.

$$\text{We have: } \sqrt{a^2 + 3bc} - a = \frac{(\sqrt{a^2 + 3bc} - a)(\sqrt{a^2 + 3bc} + a)}{\sqrt{a^2 + 3bc} + a} = \frac{3bc}{p} \text{ (and analogs)}$$

$$\text{Thus, it suffices to prove that: } \sum_{cyc} \frac{3bc}{p} \geq \sqrt{3(ab + bc + ca)} \Leftrightarrow \sum_{cyc} \frac{bc}{p} \geq \sqrt{\frac{ab + bc + ca}{3}}$$

$$\text{Now, } \sum_{cyc} \frac{bc}{p} \stackrel{\text{BERGSTROM}}{\geq} \frac{(ab + bc + ca)^2}{\sum bc \cdot p}.$$

$$\text{We have: } \left(\sum_{cyc} bc \cdot p \right)^2 \stackrel{\text{Cauchy}}{\geq} \left(\sum_{cyc} bc \right) \left(\sum_{cyc} bc \cdot p^2 \right)$$

$$\text{Also, } p^2 = \left(\sqrt{a^2 + 3bc} + a \right)^2 = 2a^2 + 3bc + 2a\sqrt{a^2 + 3bc}$$

$$\text{So, } \sum_{cyc} bc \cdot p^2 = 2abc(a + b + c) + 3 \sum_{cyc} b^2c^2 + 2abc \sum_{cyc} \sqrt{a^2 + 3bc}$$

$$\sum_{cyc} \sqrt{a^2 + 3bc} \stackrel{\text{Cauchy}}{\geq} \sqrt{3 \sum_{cyc} (a^2 + 3bc)} =$$

$$= \sqrt{3 \left(\sum_{cyc} a \right)^2 + 3 \sum_{cyc} ab} \stackrel{3 \sum ab \leq (\sum a)^2}{\geq} \sqrt{3 \left(\sum_{cyc} a \right)^2 + \left(\sum_{cyc} a \right)^2} = 2 \sum_{cyc} a$$

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$$\therefore \sum_{cyc} bc \cdot p^2 \leq 2abc(a+b+c) + 3 \sum_{cyc} b^2c^2 + 2abc \cdot 2 \sum_{cyc} a = 3(ab+bc+ca)^2$$

$$\therefore \left(\sum_{cyc} bc \cdot p \right)^2 \stackrel{\text{Cauchy}}{\leq} (ab+bc+ca) \cdot 3(ab+bc+ca)^2 = 3(ab+bc+ca)^3$$

$$\sum_{cyc} bc \cdot p \leq \sqrt{3(ab+bc+ca)^3}$$

$$\sum_{cyc} \frac{bc}{p} \stackrel{\text{BERGSTROM}}{\geq} \frac{(ab+bc+ca)^2}{\sum bc \cdot p} \geq \frac{(ab+bc+ca)^2}{\sqrt{3(ab+bc+ca)^3}} = \sqrt{\frac{ab+bc+ca}{3}}$$

Solution to problem 2:

Dividing by abc , we get,

$$\sqrt{\left(\frac{a^2}{bc} + 3\right)\left(\frac{b^2}{ca} + 3\right)\left(\frac{c^2}{ab} + 3\right)} \geq \left(\frac{a}{\sqrt{bc}} + 1\right)\left(\frac{b}{\sqrt{ca}} + 1\right)\left(\frac{c}{\sqrt{ab}} + 1\right)$$

Let $x = \frac{a}{\sqrt{bc}}$, $y = \frac{b}{\sqrt{ca}}$ and $z = \frac{c}{\sqrt{ab}}$. Notice: $xyz = 1$

It suffices to prove that: $(x^2 + 3)(y^2 + 3)(z^2 + 3) \geq [(x+1)(y+1)(z+1)]^2$

Let $p = x + y + z$ and $q = xy + yz + zx$. Since $xyz = 1$, we have,

$$\begin{aligned} (x^2 + 3)(y^2 + 3)(z^2 + 3) &= x^2y^2z^2 + 3(x^2y^2 + y^2z^2 + z^2x^2) + 9(x^2 + y^2 + z^2) + 27 \\ &= 1 + 3(q^2 - 2p) + 9(p^2 - 2q) + 27 \\ &= 9p^2 + 3q^2 - 6p - 18q + 28 \end{aligned}$$

$$\begin{aligned} [(x+1)(y+1)(z+1)]^2 &= [xyz + (xy + yz + zx) + (x + y + z) + 1]^2 \\ &= [p + q + 2]^2 = p^2 + q^2 + 2pq + 4p + 4q + 4 \end{aligned}$$

$$\begin{aligned} &(9p^2 + 3q^2 - 6p - 18q + 28) - (p^2 + q^2 + 2pq + 4p + 4q + 4) = \\ &= 2(4p^2 + q^2 - pq - 5p - 11q + 12) = 2 \cdot \left(\frac{p-2q+3}{2}\right)^2 + \frac{13}{6}(p-3)^2 + \frac{16}{3}(p^2 - 3q) \end{aligned}$$

, which is ≥ 0 as $p^2 = (x+y+z)^2 \geq 3(xy+yz+zx) = 3q$.

Equality holds if and only if $a = b = c$ for both problems.