

ROMANIAN MATHEMATICAL MAGAZINE

*Let a, b, c, d be positive real numbers satisfying
 $a > b > c > d$. Find maximum value of*

$$P = \frac{(a-b)(a-c)(b-c)(b-d)(c-d)}{(a-d)^5}$$

Proposed by Le Anh Tu-Vietnam

Solution by Tapas Das-India

Let $a' = a - d, b' = b - d, c' = c - d$ & $a > b > c > d \Rightarrow a' > b' > c' > 0$

Let $x = \frac{b'}{a'}, y = \frac{c'}{a'}$, since $a' > b' > c' > 0$ then $1 > x > y > 0$

using above substitution we get $P = \frac{(a' - b')(a' - c')(b' - c')b'c'}{(a')^5}$

now we take $f(x, y) = P \stackrel{x=\frac{b'}{a'}, y=\frac{c'}{a'}}{=} (1-x)(1-y)(x-y)xy$

Let $g(x, y) = \ln f(x, y) = \ln x + \ln y + \ln(x-y) + \ln(1-x) + \ln(1-y)$

$$g_x = \frac{\delta g}{\delta x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{x-y} = \frac{1}{x-y} + \frac{1-2x}{x(1-x)}, g_{xx} = \frac{\delta^2 g}{\delta x^2}$$

$$= -\frac{1}{x^2} - \frac{1}{(1-x)^2} - \frac{1}{(x-y)^2}$$

$$g_y = \frac{\delta g}{\delta y} = \frac{1}{y} - \frac{1}{1-y} - \frac{1}{x-y} = \frac{1-2y}{y(1-y)} - \frac{1}{x-y}, g_{yy} = \frac{\delta^2 g}{\delta y^2}$$

$$= -\frac{1}{y^2} - \frac{1}{(1-y)^2} - \frac{1}{(x-y)^2}$$

$g_{xy} = \frac{\delta}{\delta x} \left(\frac{\delta g}{\delta y} \right) = \frac{1}{(x-y)^2}$ now we find critical point, for this $g_x = 0$ & $g_y = 0$

$$\frac{1}{x-y} + \frac{1-2x}{x(1-x)} = 0 \text{ or } \frac{1}{x-y} = \frac{2x-1}{x(1-x)} \text{ \& } \frac{1-2y}{y(1-y)} - \frac{1}{x-y} = 0 \text{ or}$$

$$\frac{1}{x-y} = \frac{1-2y}{y(1-y)}$$

so we have $\frac{2x-1}{x(1-x)} = \frac{1-2y}{y(1-y)}$ or, $x(1-x)(1-2y) = y(1-y)(2x-1)$

or, $x - x^2 - 2xy + 2x^2y$

$$= -y + y^2 + 2x(y - y^2) \text{ or, } (x+y-1)(x(y-1) + y(x-1)) = 0$$

so $x+y-1=0$ (since $x, y < 1$) or, $x+y=1$ or, $x=1-y$

from $\frac{1}{x-y} = \frac{2x-1}{x(1-x)}$ we get $\frac{1}{1-y-y} = \frac{2(1-y)-1}{(1-y)y}$ or, $y - y^2 = 1 - 4y + 4y^2$

ROMANIAN MATHEMATICAL MAGAZINE

$$\text{or, } 5y^2 - 5y + 1 = 0 \text{ or, } y = \frac{5 \pm \sqrt{5}}{10} \text{ as } x + y = 1 \text{ \& } 1 > x > y \text{ so } x > \frac{1}{2}$$

$$\text{so we can take } y = \frac{5 - \sqrt{5}}{10} = b, x = 1 - y = \frac{5 + \sqrt{5}}{10} = a$$

$$\begin{aligned} \text{now } g_{xx} \text{ at } (a, b) &= \left(-\frac{1}{x^2} - \frac{1}{(1-x)^2} - \frac{1}{(x-y)^2} \right)_{(a,b)} \\ &= \left(-\left(\frac{1}{x^2} + \frac{1}{y^2} \right) - \frac{1}{(1-2y)^2} \right)_{a,b} \\ &= \left(-\frac{x^2 + y^2}{x^2 y^2} - \frac{1}{(1-2y)^2} \right)_{a,b} = -15 - 5 = -20 \text{ (putting value of } a \text{ \& } b) \end{aligned}$$

$$\text{similarly, } g_{yy} \text{ at } (a, b) = -20 \text{ \& } g_{xy} \text{ at } (a, b) = 5$$

now maxma minima for two variable

$$D = g_{xx}g_{yy} - (g_{xy})^2 = (-20)(-20) - (5)^2 = 375 > 0 \text{ and } g_{xx} = -20 < 0$$

implies $g(x, y)$ or $f(x, y)$ maximum at (a, b) where $x = 1 - y$

$$\begin{aligned} P &= (1-x)(1-y)(x-y)xy \text{ or, } P_{\max.}^{x=1-y} = y \cdot x \cdot (x-y) \cdot xy = (xy)^2(x-y) \\ &= (ab)^2(a-b) = \left(\frac{5+\sqrt{5}}{10} \cdot \frac{5-\sqrt{5}}{10} \right)^2 \left(\frac{5+\sqrt{5}}{10} - \frac{5-\sqrt{5}}{10} \right) = \frac{1}{25} \cdot \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{125} \\ \text{at } a = x &= \frac{5+\sqrt{5}}{10}, b = y = \frac{5-\sqrt{5}}{10} \end{aligned}$$