

ROMANIAN MATHEMATICAL MAGAZINE

Let a, b, c be positive real numbers. Prove the following inequalities:

$$\sqrt{a^2 + ab + b^2} + \sqrt{b^2 + bc + c^2} + \sqrt{c^2 + ca + a^2} \geq \sqrt{4(a^2 + b^2 + c^2) + 5(ab + bc + ca)}$$

$$\sqrt{a^2 + ab + b^2} + \sqrt{b^2 + bc + c^2} + \sqrt{c^2 + ca + a^2} \leq \sqrt{5(a^2 + b^2 + c^2) + 4(ab + bc + ca)}$$

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Solution for problem 1:

$$\sum_{cyc} \sqrt{a^2 + ab + b^2} \geq \sqrt{4 \sum_{cyc} a^2 + 5 \sum_{cyc} ab}$$

$$\stackrel{\text{Squaring}}{\Leftrightarrow} \sum_{cyc} (a^2 + ab + b^2) + 2 \sum_{cyc} \left(\sqrt{a^2 + ab + b^2} \cdot \sqrt{a^2 + ac + c^2} \right) \stackrel{?}{\geq}$$

$$\geq 4 \sum_{cyc} a^2 + 5 \sum_{cyc} ab$$

$$\Leftrightarrow 2 \sum_{cyc} \left(\sqrt{a^2 + ab + b^2} \cdot \sqrt{a^2 + ac + c^2} \right) \stackrel{?}{\geq} 4 \sum_{cyc} a^2 + 5 \sum_{cyc} ab - \sum_{cyc} (a^2 + ab + b^2)$$

Thus, it suffices to prove that: $\sum_{cyc} \left(\sqrt{a^2 + ab + b^2} \cdot \sqrt{a^2 + ac + c^2} \right)$

$$\geq \sum_{cyc} a^2 + 2 \sum_{cyc} ab$$

Notice that: $a^2 + ab + b^2 = \frac{1}{2} [a^2 + b^2 + (a + b)^2]$ (and analogs)

$$\begin{aligned} & \sum_{cyc} \left(\sqrt{a^2 + ab + b^2} \cdot \sqrt{a^2 + ac + c^2} \right) = \\ &= \sum_{cyc} \left(\sqrt{\frac{1}{2} [a^2 + b^2 + (a + b)^2]} \cdot \sqrt{\frac{1}{2} [a^2 + c^2 + (a + c)^2]} \right) = \\ &= \sum_{cyc} \left(\frac{1}{2} \sqrt{a^2 + b^2 + (a + b)^2} \cdot \sqrt{a^2 + c^2 + (a + c)^2} \right) \stackrel{\text{Cauchy}}{\geq} \\ &\geq \frac{1}{2} \sum_{cyc} [a(a) + bc + (a + b)(a + c)] = \\ &= \frac{1}{2} \sum_{cyc} [2a^2 + ab + 2bc + ac] = \sum_{cyc} a^2 + 2 \sum_{cyc} ab \end{aligned}$$

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Solution for problem 2:

$$\left(\sum_{cyc} \sqrt{a^2 + ab + b^2} \right)^2 = \left(\sum_{cyc} \sqrt{a+b} \cdot \sqrt{\frac{a^2 + ab + b^2}{a+b}} \right)^2 \stackrel{\text{Cauchy}}{\leq} \left(\sum_{cyc} (a+b) \right) \left(\sum_{cyc} \frac{a^2 + ab + b^2}{a+b} \right)$$

$$\text{Also, } \sum_{cyc} \frac{a^2 + ab + b^2}{a+b} = \sum_{cyc} (a+b) - \frac{ab}{a+b} = 2(a+b+c) - \sum_{cyc} \frac{ab}{a+b}$$

$$\begin{aligned} \therefore \left(\sum_{cyc} \sqrt{a^2 + ab + b^2} \right)^2 &\leq 2(a+b+c) \left[2(a+b+c) - \sum_{cyc} \frac{ab}{a+b} \right] = \\ &= 4(a+b+c)^2 - \sum_{cyc} \frac{2ab(a+b+c)}{a+b} \end{aligned}$$

$$\begin{aligned} \sum_{cyc} \frac{2ab(a+b+c)}{a+b} &= \sum_{cyc} 2ab \left(1 + \frac{c}{a+b} \right) = 2(ab+bc+ca) + \sum_{cyc} \frac{2abc}{a+b} = \\ &= 2(ab+bc+ca) + 2abc \sum_{cyc} \frac{1}{a+b} \end{aligned}$$

$$\therefore \left(\sum_{cyc} \sqrt{a^2 + ab + b^2} \right)^2 \leq 4(a+b+c)^2 - 2(ab+bc+ca) - 2abc \sum_{cyc} \frac{1}{a+b}$$

By Titu's inequality, we have: $\sum_{cyc} \frac{1}{a+b} \geq \frac{9}{2(a+b+c)}$. So,

$$\left(\sum_{cyc} \sqrt{a^2 + ab + b^2} \right)^2 \leq 4(a+b+c)^2 - 2(ab+bc+ca) - \frac{9abc}{a+b+c}$$

It suffices to prove that:

$$4(a+b+c)^2 - 2(ab+bc+ca) - \frac{9abc}{a+b+c} \leq 5(a^2+b^2+c^2) + 4(ab+bc+ca)$$

$$\Leftrightarrow a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \geq 2(ab+bc+ca),$$

which is true as it is **Schur's inequality** of degree 3.

Equality holds iff and only $a = b = c$.