

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\sum_{cyc} \frac{xy(x^2 - xy + y^2)}{x^2 + xy + y^2} \leq \frac{2}{3} \sum_{cyc} \frac{x^4}{x^2 + yz}$$

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$$\text{Lemma: } \forall a, b > 0, \frac{ab(a^2 - ab + b^2)}{a^2 + ab + b^2} \stackrel{AM-GM}{\leq} \frac{ab(a^2 - ab + b^2)}{3ab} = \frac{1}{3}(a^2 - ab + b^2)$$

$$L.H.S = \sum_{cyc} \frac{xy(x^2 - xy + y^2)}{x^2 + xy + y^2} \stackrel{\text{lemma}}{\leq} \frac{1}{3} \sum (x^2 - xy + y^2) = \frac{2}{3} \sum x^2 - \frac{1}{3} \sum xy$$

$$R.H.S = \frac{2}{3} \sum_{cyc} \frac{x^4}{x^2 + yz} = \frac{2}{3} \sum \left(x^2 - \frac{x^2 yz}{x^2 + yz} \right) \stackrel{AM-GM}{\leq} \frac{2}{3} \sum x^2 - \frac{2}{3} \sum \frac{x^2 yz}{2x\sqrt{yz}} =$$

$$= \frac{2}{3} \sum x^2 - \frac{1}{3} \sum x\sqrt{yz} \stackrel{AM-GM}{\leq} \frac{2}{3} \sum x^2 - \frac{1}{3} \sum \frac{x(y+z)}{2} = \frac{2}{3} \sum x^2 - \frac{1}{3} \sum xy$$

so $L.H.S \leq R.H.S.$

Equality holds for $x = y = z$