

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\frac{a+b+c}{\sqrt[3]{abc}} \geq 1 + \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}$$

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Let $x = (a+b+c), y = \sqrt[3]{abc}$ then by AM - GM $x \geq 3y$ (1)

We will show:

$$\left((a+b+c) - \sqrt[3]{abc}\right)^3 \geq (a+b)(b+c)(c+a)$$

Proof: $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc \leq$

$$\leq (a+b+c) \frac{(a+b+c)^2}{3} - abc \stackrel{a+b+c=x}{=} \frac{(abc)^{\frac{1}{3}}=y}{3} x^3 - y^3 = \frac{x^3 - 3y^3}{3}$$

$$\left((a+b+c) - \sqrt[3]{abc}\right)^3 = (x-y)^3 = x^3 - y^3 - 3xy(x-y)$$

we need to show : $x^3 - y^3 - 3xy(x-y) \geq \frac{x^3 - 3y^3}{3}$ or, $2x^3 \geq 9xy(x-y)$

or, $2x^2 - 9xy + 9y^2 \geq 0$ or, $(x-3y)(2x-3y) \geq 0$ True by (1)

$$\frac{a+b+c}{\sqrt[3]{abc}} = 1 + \left(\frac{a+b+c}{\sqrt[3]{abc}} - 1\right) = 1 + \frac{a+b+c - \sqrt[3]{abc}}{\sqrt[3]{abc}} =$$

$$= 1 + \sqrt[3]{\frac{\left((a+b+c) - \sqrt[3]{abc}\right)^3}{abc}} \geq 1 + \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}$$

Equality holds for $a=b=c$