

ROMANIAN MATHEMATICAL MAGAZINE

Let a, b, c be positive real numbers & $\lambda \geq 1$. Prove that:

$$\frac{a}{\lambda\sqrt{a^2+8bc}-(\lambda-1)a} + \frac{b}{\lambda\sqrt{b^2+8ac}-(\lambda-1)b} + \frac{c}{\lambda\sqrt{c^2+8ab}-(\lambda-1)c} \geq \frac{3}{2\lambda+1}$$

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$$a^2 < a^2 + 8bc \ (a, b, c > 0) \text{ or, } a < \sqrt{a^2 + 8bc} \text{ or, } \frac{a}{\sqrt{a^2 + 8bc}} < 1$$

$$\text{Let } x = \frac{a}{\sqrt{a^2 + 8bc}}, y = \frac{b}{\sqrt{b^2 + 8ac}}, z = \frac{c}{\sqrt{c^2 + 8ab}}, x, y, z \in (0, 1)$$

A famous inequality from IMO 2001 (problem 2), states that for any positive real number a, b, c we get:

$$x + y + z = \frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ac}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1 \quad (1)$$

$$\begin{aligned} & \frac{a}{\lambda\sqrt{a^2+8bc}-(\lambda-1)a} + \frac{b}{\lambda\sqrt{b^2+8ac}-(\lambda-1)b} + \frac{c}{\lambda\sqrt{c^2+8ab}-(\lambda-1)c} \\ &= \frac{\frac{a}{\sqrt{a^2+8bc}}}{\lambda - (\lambda-1)\left(\frac{a}{\sqrt{a^2+8bc}}\right)} + \frac{\frac{b}{\sqrt{b^2+8ac}}}{\lambda - (\lambda-1)\left(\frac{b}{\sqrt{b^2+8ac}}\right)} + \frac{\frac{c}{\sqrt{c^2+8ab}}}{\lambda - (\lambda-1)\left(\frac{c}{\sqrt{c^2+8ab}}\right)} \\ &= \frac{x}{\lambda - (\lambda-1)x} + \frac{y}{\lambda - (\lambda-1)y} + \frac{z}{\lambda - (\lambda-1)z} \end{aligned}$$

$$\text{Let } f(t) = \frac{t}{\lambda - (\lambda-1)t}, t \in (0, 1) \quad f'(t) = \frac{\lambda}{(\lambda - (\lambda-1)t)^2} > 0 \text{ \&}$$

$$f''(t) = \frac{2\lambda(\lambda-1)}{(\lambda - (\lambda-1)t)^3} > 0 \text{ (as } \lambda - (\lambda-1)t = \lambda(1-t) + t > 0, t < 1 \text{ and } \lambda > 1)$$

so f is convex and increasing on $(0, 1)$,

$$\text{using Jensen inequality } f(x) + f(y) + f(z) \geq 3f\left(\frac{x+y+z}{3}\right) \stackrel{(1)}{\geq} 3f\left(\frac{1}{3}\right)$$

$$\begin{aligned} & \frac{a}{\lambda\sqrt{a^2+8bc}-(\lambda-1)a} + \frac{b}{\lambda\sqrt{b^2+8ac}-(\lambda-1)b} + \frac{c}{\lambda\sqrt{c^2+8ab}-(\lambda-1)c} \geq \\ & \geq 3 \frac{\frac{1}{3}}{\lambda - (\lambda-1)\frac{1}{3}} = \frac{3}{2\lambda+1} \end{aligned}$$

Equality holds for $a=b=c$.