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$$If \begin{cases} a_1, a_2, \dots, a_n \in R^+ \\ b_1, b_2, \dots, b_n \in R^+ \end{cases} and\ p, q > 0\ \frac{1}{p} + \frac{1}{q} = 1\ then\ prove$$

$$\left[\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} a_i\right)^2 + \frac{n^2}{4}\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}\right] \sum_{i=1}^{n} \frac{1}{a_i} \ge n^3 \sqrt{\sum_{i=1}^{n} a_i b_i}$$

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$$LHS = \left[\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} a_i\right)^2 + \frac{n^2}{4}\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}\right] \sum_{i=1}^{n} \frac{1}{a_i} = [A+B] \sum_{i=1}^{n} \frac{1}{a_i}$$

$$A + B \overset{AM-GM}{\ge} 2\sqrt{AB} \to A + B \ge 2\sqrt{\left[\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} a_i\right)^2 \frac{n^2}{4}\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}\right]} =$$

$$= n\sqrt{\left[\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} a_i\right)^2 \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}\right]} = n\sum_{i=1}^{n} a_i \sqrt{\left[\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}\right]}$$

$$0 < p, q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \left[\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}\right] \overset{Holders}{\ge} \sum_{i=1}^{n} a_i b_i$$

$$A + B \ge n\sum_{i=1}^{n} a_i \sqrt{\left[\sum_{i=1}^{n} a_i b_i\right]} \to LHS \ge [A+B] \sum_{i=1}^{n} \frac{1}{a_i}$$

$$LHS \ge n\sum_{i=1}^{n} a_i \sum_{i=1}^{n} \frac{1}{a_i} \sqrt{\left[\sum_{i=1}^{n} a_i b_i\right]} \to \sum_{i=1}^{n} a_i \sum_{n=1}^{n} \frac{1}{a_i} \overset{Bunyakovski}{\ge} n^2$$

$$LHS \geq n^3 \sqrt{\sum_{i=1}^{n} a_i b_i} \;\; Equality\ holds\ for \begin{cases} a_1 = a_2 = \cdots = a_n \\ b_1 = b_2 = \cdots = b_n \end{cases}$$

$$\left[\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} a_i\right)^2 + \frac{n^2}{4}\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}\right]\sum_{i=1}^{n}\frac{1}{a_i} \geq n^3 \sqrt{\sum_{i=1}^{n} a_i b_i} \quad (Q.E.D)$$