ROMANIAN MATHEMATICAL MAGAZINE

Let $a_1, a_2, a_3, a_4,$ be positive real numbers $a_1.a_2.a_3.a_4 = 1$. Prove that :

$$\sum_{cyc} \frac{1}{a_1^3(a_2+a_3+a_4)} \geq \frac{4}{3}$$

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Solution by Khaled Abd Imouti-Syria

$$LHS = \frac{1}{a_1^3(a_2+a_3+a_4)} + \frac{1}{a_2^3(a_1+a_3+a_4)} + \frac{1}{a_3^3(a_1+a_2+a_4)} + \frac{1}{a_4^3(a_1+a_2+a_3)}$$

$$LHS = \frac{\frac{1}{a_1^2}}{a_1(a_2+a_3+a_4)} + \frac{\frac{1}{a_2^2}}{a_2(a_1+a_3+a_4)} + \frac{\frac{1}{a_3^2}}{a_3(a_1+a_2+a_4)} + \frac{\frac{1}{a_4^2}}{a_4(a_1+a_2+a_3)}$$

$$LHS = \frac{(\frac{1}{a_1})^2}{a_1(a_2+a_3+a_4)} + \frac{(\frac{1}{a_2})^2}{a_2(a_1+a_3+a_4)} + \frac{(\frac{1}{a_3})^2}{a_3(a_1+a_2+a_4)} + \frac{(\frac{1}{a_4})^2}{a_4(a_1+a_2+a_3)}$$

Suppose : $x = \frac{1}{a_1}$, $y = \frac{1}{a_2}$, $z = \frac{1}{a_3}$, $t = \frac{1}{a_4}$

$$LHS = \frac{x^2}{\frac{1}{xy}+\frac{1}{xz}+\frac{1}{xt}} + \frac{y^2}{\frac{1}{yx}+\frac{1}{yz}+\frac{1}{yt}} + \frac{z^2}{\frac{1}{zx}+\frac{1}{zy}+\frac{1}{zt}} + \frac{t^2}{\frac{1}{tx}+\frac{1}{ty}+\frac{1}{tz}}$$

$$LHS = \frac{x^3yzt}{zt+yt+zy} + \frac{y^3xzt}{zt+xt+xz} + \frac{z^3xyt}{ty+xt+xy} + \frac{t^3xyz}{zy+zx+xy}$$

$$LHS = \frac{x^2}{zt+yt+zy} + \frac{y^2}{zt+xt+xz} + \frac{z^2}{ty+xt+xy} + \frac{t^2}{zy+zx+xy}$$

$$LHS \geq \frac{(x+y+z+t)^2}{2(xy+xz+xt+yz+yt+zt)} \quad \textit{by Bergstrom's inequality}$$

But : $(x+y+z+t)^2 \geq \frac{2}{3}(xy+xz+xt+yz+yt+zt) \Rightarrow$ *So*: $L \geq \frac{\frac{2}{3}}{2} = \frac{4}{3}$