ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, a^3+b^3+c^3=ab+ac+bc$ then:

$$\frac{a^6}{b^2c^2}+\frac{b^6}{a^2c^2}+\frac{c^6}{a^2b^2}\ge\frac{9abc}{a+b+c}$$

Proposed by Gheorghe Crăciun-Romania

Solution by Amin Hajiyev-Azerbaijan

$$\frac{\frac{a^6}{b^2c^2}+b^2+c^2}{3}\overset{AM-GM}{\ge}\sqrt[3]{\frac{a^6}{b^2c^2}*b^2*c^2}=a^2$$

$$\frac{a^6}{b^2c^2}+b^2+c^2\ge 3a^2\longrightarrow\sum_{cyc}\left(\frac{a^6}{b^2c^2}+b^2+c^2\right)\ge 3\sum_{cyc}a^2$$

$$\frac{a^6}{b^2c^2}+\frac{b^6}{a^2c^2}+\frac{c^6}{a^2b^2}+2(a^2+b^2+c^2)\ge 3(a^2+b^2+c^2)$$

$$\sum_{cyc}\frac{a^6}{b^2c^2}\ge a^2+b^2+c^2$$

$$a+b+c\overset{AM-GM}{\ge}3\sqrt[3]{abc},\quad a^2+b^2+c^2\overset{AM-GM}{\ge}3\sqrt[3]{a^2b^2c^2}$$

$$(a+b+c)(a^2+b^2+c^2)\ge 9abc\rightarrow\quad a^2+b^2+c^2\ge\frac{9abc}{a+b+c}$$

$$\sum_{cyc}\frac{a^6}{b^2c^2}\ge a^2+b^2+c^2\ge\frac{9abc}{a+b+c}$$

$$\frac{a^6}{b^2c^2}+\frac{b^6}{a^2c^2}+\frac{c^6}{a^2b^2}\ge\frac{9abc}{a+b+c}$$

Equality holds for $a=b=c=1$.