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ON THE REPRESENTATION FORMULA OF A NEWLY TRANSFORMED FRACTIONAL-ORDER BESSEL FUNCTION

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Abstract: In this paper a formula for the fractional-order Bessel function is obtained at a new level of interpretation, which distinguishes it from the formulas previously reported in the literature.

***Key words and Phrases*:** Bessel function, gamma function, double factorial

1. Introduction

In this article, a known representation formula for the fractional-order Bessel function is derived by introducing the concept of the double factorial. Subsequently, by applying suitable approximation transformations to this formula, a new form of the representation is obtained. The resulting expression provides an alternative analytical structure for the fractional-order Bessel function, which may simplify further theoretical analysis and computational implementations.

2 .Main formula

The article contains a number of formulas that are considered fundamental, and these formulas are rigorously proven within the framework of the study.

$$\int_0^{\frac{\pi}{2}} sin^{2k}\, x\ dx\ = \frac{\pi}{2}\,\frac{(2k-1)!!}{(2k)!!}$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sum_{k=0}^{\infty}\frac{(-1)^k\ x^{2k+1}}{2^k k!\ \frac{2}{\pi}\ (2k+2)!!\int_0^{\frac{\pi}{2}} sin^{2k+2}\, x\ dx}$$

3. Main problem and solution :

Here, in proving the representation formula of the fractional-order Bessel function, several technically different methods have been employed. Among these methods are the use of the double factorial concept and the application of certain formulas related to the Gamma function. The intermediate steps and transitions are explained as follows:

$$J_{\frac{1}{2}}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\ \left(\ \frac{x}{2}\ \right)^{2k+\frac{1}{2}}}{k!\,\Gamma\left(k+\frac{3}{2}\right)} =$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\ \frac{x^{2k+1}}{2^{2k+\frac{1}{2}}}\ x^{-\frac{1}{2}}}{k!\,\Gamma\left(k+\frac{3}{2}\right)} = \frac{1}{\sqrt{x}}\sum_{k=0}^{\infty}\frac{(-1)^k\ x^{2k+1}}{k!\,\Gamma\left(k+\frac{3}{2}\right)2^{2k+\frac{1}{2}}} =$$

$$= \frac{\sqrt{2}}{\sqrt{x}}\sum_{k=0}^{\infty}\frac{(-1)^k\ x^{2k+1}}{k!\,\Gamma\left(k+\frac{3}{2}\right)2^k 2^k 2} =$$

$$= \frac{\sqrt{2}}{\sqrt{x}}\sum_{k=0}^{\infty}\frac{(-1)^k\ x^{2k+1}}{k!\frac{(2k+1)!!}{2^{k+1}}\sqrt{\pi}\ 2^k 2^k 2} = \quad \frac{\sqrt{2}}{\sqrt{x}}\sum_{k=0}^{\infty}\frac{(-1)^k\ x^{2k+1}}{k!\frac{(2k+1)!!}{2^{k+1}}\sqrt{\pi}\ 2^k 2^k 2} \quad (1)$$

(1) By applying several technical identities to formula (1), we obtain the following result:

$$(2k-1)!! = \frac{(2k)!}{2^k k!} \qquad (2k+1)!! = \frac{(2k+2)!}{2^{k+1}(k+1)!}$$

$$\frac{\sqrt{2}}{\sqrt{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{2k+1}}{k!\, 2^k \frac{(2k+2)!}{2^{k+1}(k+1)!}} = \frac{\sqrt{2}}{\sqrt{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{2k+1}}{k!\, 2^k \frac{(2k+2)(2k+1)!}{2^{k+1}(k+1)!}} =$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{2k+1}}{(2k+1)!} = \sqrt{\frac{2}{\pi x}} sinx$$

Let us consider the following integral:

$$\int_0^{\frac{\pi}{2}} sin^n\, x \ dx \ == \int_0^{\frac{\pi}{2}} sin^{n-1}\, x\, sinx\, dx =$$

$$= \int_0^{\frac{\pi}{2}} (sin^2\, x)^{\frac{n-1}{2}} sinx\, dx = \int_0^{\frac{\pi}{2}} (1 - cos^2\, x)^{\frac{n-1}{2}} sinx\, dx$$

→ By performing the substitution cosx=t and, furthermore, incorporating characteristic formulas of the Gamma function, we obtain the following equality, that is, equivalently, the following formula:

$$\int_0^{\frac{\pi}{2}} sin^{2k}\, x \ dx \ = \frac{\pi}{2} \frac{(2k-1)!!}{(2k)!!}$$

By replacing k with k+1, we obtain the following expression for (2k+1)!!

$$(2k+1)!! = \frac{2}{\pi} \ (2k+2)!! \int_0^{\frac{\pi}{2}} sin^{2k+2}\, x \ dx$$

Hence, we obtain the following formula for the fractional-order Bessel function:

$$J_{\frac{1}{2}}(x) \ = \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{2k+1}}{2^k k!\, (2k+1)!!} =$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{2k+1}}{2^k k!\, \frac{2}{\pi} \ (2k+2)!! \int_0^{\frac{\pi}{2}} sin^{2k+2}\, x \ dx} \qquad (2)$$

4 . Conclusion

The formulas (1)–(2) obtained herein are referred to as the representation formulas of the transformed fractional-order Bessel function. Thus, instead of explicitly writing several terms of the series representation of the fractional-order Bessel function and demonstrating that it corresponds to the power series of a trigonometric function, we employ an alternative technique to prove that it can be expressed in the form of the power series of the hyperbolic sine function.

In the first representation formula of the transformed fractional-order Bessel function, by replacing the expression involving odd terms—through the use of the double factorial concept—with its corresponding integral representation, the second representation formula of the transformed fractional-order Bessel function is derived. Finally, we present the representation formulas of the transformed Bessel function in a generalized form as follows:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{2k+1}}{2^k k!\,(2k+1)!!}$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k \ x^{2k+1}}{2^k k!\ \frac{2}{\pi}\ (2k+2)!! \int_0^{\frac{\pi}{2}} sin^{2k+2}\, x \ dx}$$