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PROBLEMS FOR JUNIORS

JP.586 If a, b, c are the sides of acute triangle ABC and

$$\frac{2}{a^2+b^2-b^2}=\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+b^2-c^2}$$

then $\tan^2 B \geq \tan A \cdot \tan C$.

Proposed by Daniel Sitaru – Romania

Solution 1 by Jose Luis Diaz – Barrero – Spain

If ABC is an acute triangle with side lengths a, b, c opposite angles A, B, C, then the cosine rule gives:

$$b^2+c^2-a^2=2bc\cos A$$
$$a^2+c^2-b^2=2ac\cos B$$
$$a^2+b^2-c^2=2ab\cos C$$

Substituting these into the given identity yields

$$\frac{1}{ac\cos B}=\frac{1}{2bc\cos A}+\frac{1}{2ab\cos C} \qquad (1)$$

Multiplying both sides of (1) by $2abc\cos A\cos B\cos C$ (all positive since the triangle is acute) gives

$$2b\cos A\cos C=a\cos B\cos C+c\cos A\cos B \qquad (2)$$

Now apply the sine rule $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$:

$$4R\sin B\cos A\cos C=2R\sin A\cos B\cos C+2R\sin C\cos A\cos B.$$

Dividing by $2R\cos A\cos B\cos C$, we get

$$\frac{2\sin B}{\cos B}=\frac{\sin A}{\cos A}+\frac{\sin C}{\cos C}.$$

Thus,

$$2\tan B=\tan A+\tan C. \qquad (3)$$

We now prove the inequality claimed. Using (3),

$$\tan B=\frac{\tan A+\tan C}{2},$$

so

$$\tan^2 B = \left(\frac{\tan A + \tan C}{2}\right)^2 \geq \tan A \tan C,$$

which is equivalent to

$$(\tan A - \tan C)^2 \geq 0.$$

Therefore,

$$\tan^2 B \geq \tan A \tan C.$$

Equality holds precisely when $A = C$.

Solution 2 by proposer

$$\frac{2}{a^2+c^2-b^2} = \frac{1}{b^2+c^2-a^2} + \frac{1}{c^2+a^2-b^2}$$

Multiplying with $4F$; (F – area):

$$\frac{8F}{a^2+c^2-b^2} = \frac{4F}{b^2+c^2-a^2} + \frac{4F}{c^2+a^2-b^2}$$

$$\frac{2}{\frac{a^2+c^2-b^2}{4F}} = \frac{1}{\frac{b^2+c^2-a^2}{4F}} + \frac{1}{\frac{c^2+a^2-b^2}{4F}}$$

$$\frac{2}{\frac{a^2+c^2-b^2}{2ac\sin B}} = \frac{1}{\frac{b^2+c^2-a^2}{2bc\sin A}} + \frac{1}{\frac{c^2+a^2-b^2}{2ca\sin B}}$$

$$\frac{2}{\frac{\cos B}{\sin B}} = \frac{1}{\frac{\cos A}{\sin A}} + \frac{1}{\frac{\cos B}{\sin B}}$$

$$\frac{2}{\cot B} = \frac{1}{\cot A} + \frac{1}{\cot B}$$

$$2\tan B = \tan A + \tan C \overset{AM-GM}{\geq} 2\sqrt{\tan A \cdot \tan C}$$

$$2\tan B \geq 2\sqrt{\tan A \cdot \tan C} \Rightarrow \tan B \geq \sqrt{\tan A \cdot \tan C}$$

$$\tan^2 B \geq \tan A \cdot \tan C$$

Equality holds for $a = c$.

JP.587 If $x,y\geq 1$ then:

$$\ln(xy)\cdot(2\ln(xy)+1)\geq 4\left(\ln x\sqrt{\ln y}+\ln y\sqrt{\ln x}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Jose Luis Diaz – Barrero – Spain

Let $a=\ln x, b=\ln y$, so that $a,b\geq 0$ because $x,y\geq 1$. Using the preceding the inequality claimed becomes

$$(a+b)(2(a+b)+1)\geq 4\left(a\sqrt{b}+b\sqrt{a}\right).$$

Now we put $u=\sqrt{a}, v=\sqrt{b}$, so that $a=u^2$ and $b=v^2$. Then the inequality to be proven is

$$(u^2+v^2)(2(u^2+v^2)+1)\geq 4uv(u+v).$$

We claim that for all $u,v\geq 0$ is $u^2+v^2\geq 2uv$, and $2(u^2+v^2)+1\geq 2(u+v)$, as can be easily proven. Indeed, $u^2+v^2\geq 2uv\Leftrightarrow (u-v)^2\geq 0$ and

$$2(u^2+v^2)+1\geq 2(u+v)\Leftrightarrow 2(u^2+v^2)+1-2(u+v)=$$

$$=2\left(u-\frac{1}{2}\right)^2+2\left(v-\frac{1}{2}\right)^2+\frac{1}{2}>0.$$

Multiplying up the two preceding inequalities yields

$$(u^2+v^2)(2(u^2+v^2)+1)\geq (2uv)\cdot 2(u+v)=4uv(u+v),$$

as desired. Equality occurs when $u=v=0$, hence $a=u^2=0, b=v^2=0$, so $x=y=1$.

Solution 2 by proposer

The inequality can be written:

$$(\ln x+\ln y)(2(\ln x+\ln y)+1)\geq 4\left(\ln x\sqrt{\ln y}+\ln y\sqrt{\ln x}\right)$$

Denote: $u=\sqrt{\ln x}\,; v=\sqrt{\ln y}$

$$x,y\geq 1\Rightarrow u,v\geq 0$$

$$(u^2+v^2)(2(u^2+v^2)+1)\geq 4\left(u^2\sqrt{v^2}+v^2\sqrt{u^2}\right)$$

$$2(u^2+v^2)^2+(u^2+v^2)\geq 4(u^2v+v^2u)$$

$$(u^2+v^2)(2(u^2+v^2)+1)\geq 4uv(u+v)\ (to\ prove)$$

$$u^2+v^2\geq 2uv\Leftrightarrow (u-v)^2\geq 0\ (1)$$

$$2(u^2+v^2)+1\ge 2(u+v)\Leftrightarrow$$

$$2(u^2+v^2)+1-2(u+v)=2\left(u-\frac{1}{2}\right)^2+2\left(v-\frac{1}{2}\right)^2+\frac{1}{2}>0$$

$$2(u^2+v^2)+1\ge 2(u+v)\ \ (2)$$

By $(1),(2)$:

$$(u^2+v^2)(2(u^2+v^2)+1)\ge 4uv(u+v)$$

Equality holds for $u=v=0\Rightarrow x=y=1$

JP.588 If $a,b,c>0, abcd=1$ then:

$$a^{b+c+d}+b^{c+d+a}c^{d+a+b}d^{a+b+c}\le 1$$

Proposed by Marin Chirciu– Romania

Solution 1 by Jose Luis Diaz – Barrero – Spain

Let us denote

$$P=a^{b+c+d}b^{c+d+a}c^{d+a+b}d^{a+b+c},$$

then

$$\ln P=(b+c+d)\ln a+(c+d+a)\ln b+(d+a+b)\ln c+(a+b+c)\ln d.$$

Note that $b+c+d=(a+b+c+d)-a$ (cyclic). Hence

$$\ln P=\sum_{cyc}\big((a+b+c+d)-x\big)\ln x=(a+b+c+d)\sum_{cyc}\ln x-\sum_{cyc}x\ln x.$$

Since $abcd=1$, we have $\sum_{cyc}\ln x=\ln(abcd)=0$, so

$$\ln P=-\sum_{cyc}x\ln x=-(a\ln a+b\ln b+c\ln c+d\ln d).$$

Thus the inequality $P\le 1$ is equivalent to

$$\ln P\le 0\Leftrightarrow -\sum x\ln x\le 0\Leftrightarrow \sum x\ \ln x\ge 0.$$

Consider the function $f:(0,+\infty)\to\mathbb{R}$ defined by $f(x)=x\ln x+1-x$. Since $f'(x)=\ln x$ and $f''(x)=\frac{1}{x}>0$ then f is convex and attains its minimum value at $x=1$. Thus, $f\ge 0$ and $x\ln x\ge x-1$ holds for all $x>0$. Applying it to a,b,c,d, gives

$$a\ln a+b\ln b+c\ln c+d\ln d\ge (a-1)+(b-1)+(c-1)+(d-1)=$$

$$= a + b + c + d - 4.$$

By AM-GM, since $abcd = 1$, we have

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd} = 1 \Rightarrow a+b+c+d \geq 4.$$

Therefore: $a\ln a + b\ln b + c\ln c + d\ln d \geq a+b+c+d-4 \geq 0.$

Hence $\sum x\ln x \geq 0$, so $\ln P \leq 0$, and $P \leq 1$, as required. Equality holds when

$$a = b = c = d = 1$$

Solution 2 by proposer

The inequality can be written equivalently: $a^{b+c+d}b^{c+d+a}c^{d+a+b}d^{a+b+c} \leq 1 \Leftrightarrow$

$$a^{a+b+c+d}b^{a+b+c+d}c^{a+b+c+d}d^{a+b+c+d} \leq a^a b^b c^c d^d \Leftrightarrow (abcd)^{a+b+c+d} \leq a^a b^b c^c d^d \Leftrightarrow$$

$\Leftrightarrow 1 \leq a^a b^b c^c d^d \Leftrightarrow \left(\frac{1}{a}\right)^a\left(\frac{1}{b}\right)^b\left(\frac{1}{c}\right)^c\left(\frac{1}{d}\right)^d \leq 1$, which follows from means inequality

Using means inequality we obtain:

$$LHS = \left(\frac{1}{a}\right)^a\left(\frac{1}{b}\right)^b\left(\frac{1}{c}\right)^c\left(\frac{1}{d}\right)^d \leq \left(\frac{\frac{1}{a}\cdot a + \frac{1}{b}\cdot b + \frac{1}{c}\cdot c + \frac{1}{d}\cdot d}{a+b+c+d}\right)^{a+b+c+d} =$$

$$= \left(\frac{1+1+1+1}{a+b+c+d}\right)^{a+b+c+d} =$$

$$\left(\frac{4}{a+b+c+d}\right)^{a+b+c+d} \overset{(1)}{\leq} \left(\frac{4}{4}\right)^{a+b+c+d} = 1 = RHS$$

where (1) $\Leftrightarrow a+b+c+d \geq 4$, true from means inequality and the condition from hypothesis $abcd = 1$. Equality holds if and only if $a = b = c = d = 1$.

JP.589 If $a, b, c > 0$ and $1 < \lambda \leq 2$ then:

$$\sum \frac{a(b+c)^2}{\lambda a + b + c} \leq \frac{a^2+b^2+c^2}{\lambda - 1}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Let } a = -x, b = -y, c = -z\ (x, y, z \geq 0) \text{ and then : } \sum_{cyc} \frac{a(b+c)^2}{\lambda a + b + c} =$$

$$\sum_{cyc}\frac{x(y+z)^2}{\lambda x+y+z}=\frac{1}{\lambda}\cdot\sum_{cyc}\frac{\big(\lambda x+y+z-(y+z)\big)(y+z)^2}{\lambda x+y+z}$$

$$=\frac{1}{\lambda}\cdot\sum_{cyc}(y+z)^2-\frac{1}{\lambda}\cdot\sum_{cyc}\frac{(y+z)^3}{\lambda x+y+z}$$

$$\overset{Holder}{\leq}\frac{2}{\lambda}\cdot\left(\sum_{cyc}x^2+\sum_{cyc}xy\right)-\frac{1}{\lambda}\cdot\frac{8\left(\sum_{cyc}x\right)^3}{3(\lambda+2)\left(\sum_{cyc}x\right)}$$

$$=\frac{2}{\lambda}\cdot\left(\sum_{cyc}x^2+\sum_{cyc}xy\right)-\frac{8}{3\lambda(\lambda+2)}\cdot\left(\sum_{cyc}x^2+2\sum_{cyc}xy\right)$$

$$=\frac{2(3\lambda+2)}{3\lambda(\lambda+2)}\cdot\left(\sum_{cyc}x^2\right)+\frac{2(3\lambda-2)}{3\lambda(\lambda+2)}\cdot\left(\sum_{cyc}xy\right)$$

$$\leq\frac{2(3\lambda+2)}{3\lambda(\lambda+2)}\cdot\left(\sum_{cyc}x^2\right)+\frac{2(3\lambda-2)}{3\lambda(\lambda+2)}\cdot\left(\sum_{cyc}x^2\right)\left(\because \lambda>1\Rightarrow\frac{2(3\lambda-2)}{3\lambda(\lambda+2)}>0\right)$$

$$=\frac{2(3\lambda+2)+2(3\lambda-2)}{3\lambda(\lambda+2)}\cdot\left(\sum_{cyc}x^2\right)=\frac{4}{\lambda+2}\cdot\left(\sum_{cyc}x^2\right)\overset{?}{\leq}\frac{1}{\lambda-1}\cdot\left(\sum_{cyc}a^2\right)$$

$$=\frac{1}{\lambda-1}\cdot\left(\sum_{cyc}x^2\right)\Leftrightarrow\left(\frac{4}{\lambda+2}-\frac{1}{\lambda-1}\right)\cdot\left(\sum_{cyc}x^2\right)\overset{?}{\leq}0$$

$$\Leftrightarrow\left(\frac{3(\lambda-2)}{(\lambda+2)(\lambda-1)}\right)\cdot\left(\sum_{cyc}x^2\right)\overset{?}{\leq}0\to\text{true}\because 1<\lambda\leq 2\text{ and }\sum_{cyc}x^2>0$$

$$\left(\begin{gathered}\because x=y=z=0\text{ is not possible as that would make }\lambda a+b+c\text{ and analogs}=0\\ \text{and so,}\sum_{cyc}x^2\neq 0\end{gathered}\right)$$

$$\therefore\sum_{cyc}\frac{a(b+c)^2}{\lambda a+b+c}\leq\frac{a^2+b^2+c^2}{\lambda-1}\ \forall a,b,c\leq 0\text{ and }1<\lambda\leq 2,$$

$$''=''\ \text{ iff } a=b=c\text{ and }\lambda=2\ (\text{QED})$$

Solution 2 by proposer

The inequality being homogeneous we can take $a+b+c=1$. We obtain:

$$\frac{a(b+c)^2}{\lambda a+b+c}=\frac{a(1-a)^2}{1+(\lambda-1)a}=\frac{1}{(\lambda-1)}a^2-\frac{2\lambda-1}{(\lambda-1)^2}a+\frac{\lambda^2}{(\lambda-1)^3}-\frac{\lambda^2}{(\lambda-1)^3}\frac{1}{a+1}$$

It follows:

$$LHS=\sum\frac{a(b+c)^2}{\lambda a+b+c}=\sum\left(\frac{1}{(\lambda-1)}a^2-\frac{2\lambda-1}{(\lambda-1)^2}a+\frac{\lambda^2}{(\lambda-1)^3}-\frac{\lambda^2}{(\lambda-1)^3}\frac{1}{a+1}\right)=$$

$$=\frac{1}{(\lambda-1)}\sum a^2-\frac{2\lambda-1}{(\lambda-1)^2}\sum a+3\frac{\lambda^2}{(\lambda-1)^3}-\frac{\lambda^2}{(\lambda-1)^3}\sum\frac{1}{a+1}=$$

$$\overset{CS}{\leq}\frac{1}{(\lambda-1)}\sum a^2-\frac{2\lambda-1}{(\lambda-1)^2}\cdot 1+3\frac{\lambda^2}{(\lambda-1)^3}-\frac{\lambda^2}{(\lambda-1)^3}\cdot\frac{(1+1+1)^2}{\sum(a+1)}=$$

$$=\frac{1}{(\lambda-1)}\sum a^2-\frac{2\lambda-1}{(\lambda-1)^2}\cdot 1+3\frac{\lambda^2}{(\lambda-1)^3}-\frac{\lambda^2}{(\lambda-1)^3}\cdot\frac{9}{1+3}=$$

$$=\frac{1}{(\lambda-1)}\sum a^2+\frac{5\lambda^2-12\lambda+4}{(\lambda-1)^3}=\frac{1}{\lambda-1}\sum a^2-\frac{(\lambda-2)(2-5\lambda)}{(\lambda-1)^3}\overset{1<\lambda\leq 2}{\leq}\frac{1}{\lambda-1}\sum a^2\ RHS$$

Equality holds if and only if $a=b=c$.

Note: For $\lambda=2$ we obtain the propose problem by Nguyen Viet Hung in Pure Inequalities 2/2022:

If $a,b,c>0$ then:

$$\sum\frac{a(b+c)^2}{2a+b+c}\leq a^2+b^2+c^2$$

Nguyen Viet Hung – Vietnam

JP.590 If $a,b,c>0$ then

$$\sum_{cyc}\frac{(b+c)^2}{2a^3+bc(b+c)}\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Proposed by Marin Chirciu– Romania

Solution 1 by Jose Luis Diaz – Barrero – Spain

Let us denote

$$T_a=\frac{(b+c)^2}{2a^3+bc(b+c)}$$

Then, we claim that $\frac{(b+c)^2}{2a^3+bc(b+c)}\leq\frac{1}{a}-\frac{(b-c)^2}{2(b^2+c^2)}$. Indeed, we call

$$\Delta=\left(\frac{1}{a}-\frac{(b+c)^2}{2a^3+bc(b+c)}\right)-\frac{(b-c)^2}{2(b^2+c^2)}=\frac{N}{D},$$

and we have to prove that $\Delta\geq 0$. The common denominator is

$$D=2(b^2+c^2)a\left(2a^3+bc(b+c)\right)\geq 0$$

and the numerator

$$N=2(b^2+c^2)\left(2a^3+bc(b+c)\right)-2a(b^2+c^2)(b+c)^2$$

$$-a\left(2a^3+bc(b+c)\right)(b-c)^2=(b-c)^2[(a^2+bc)^2+a^2(b+c)^2]\geq 0$$

Thus $\Delta=\frac{(b-c)^2\left[(a^2+bc)^2+a^2(b+c)^2\right]}{2(b^2+c^2)a\left(2a^3+bc(b+c)\right)}\geq 0$, as claimed. Summing cyclically, we obtain

$$\sum_{cyc}\frac{(b+c)^2}{2a^3+bc(b+c)}\leq\sum_{cyc}\left(\frac{1}{a}-\frac{(b-c)^2}{2(b^2+c^2)}\right).$$

The RHS equals

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\left[\frac{(b-c)^2}{2(b^2+c^2)}+\frac{(c-a)^2}{2(c^2+a^2)}+\frac{(a-b)^2}{2(a^2+b^2)}\right].$$

Each of the three subtracted terms is nonnegative, hence

$$\sum_{cyc}\frac{(b+c)^2}{2a^3+bc(b+c)}\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Equality holds when $a=b=c$.

Solution 2 by proposer

Lemma: If $a,b,c>0$ then:

$$\frac{2(b+c)^2}{2a^3+bc(b+c)}=\frac{(b+c)^3}{2a^3(b+c)+bc(b+c)^2}\overset{CS}{\leq}\frac{(b+c)^3}{2a^3(b+c)+2bc(b^2+c^2)}=$$

$$=\frac{(b+c)^3}{2c(a^3+b^3)+2b(a^3+c^3)}\overset{Holder}{\leq}\frac{b^3}{c(a^3+b^3)}+\frac{c^3}{b(a^3+c^3)}.$$

Proof.

$$LHS=\sum\frac{(b+c)^2}{2a^3+bc(b+c)}\overset{Lemma}{\leq}\sum\left[\frac{b^3}{c(a^2+b^2)}+\frac{c^3}{b(a^2+c^2)}\right]=$$

$$=\sum\left[\frac{b^3}{a(b^2+c^2)}+\frac{c^3}{a(b^2+c^2)}\right]=\sum\frac{b^3+c^3}{a(b^3+c^3)}=\sum\frac{1}{a}=RHS$$

Equality holds if and only if $a=b=c$.

JP.591 If $x,y,z>0, x+y+z=1$ and $\lambda\ge 26$ then:

$$x^4+y^4+z^4+\lambda xyz\le\frac{\lambda+1}{27}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$x^4+y^4+z^4+\lambda xyz\overset{?}{\le}\frac{\lambda+1}{27}\Leftrightarrow\sum_{cyc}x^4-\frac{1}{27}\overset{?}{\le}\lambda\left(\frac{1}{27}-xyz\right)$$

$$\Leftrightarrow\sum_{cyc}x^4-\frac{1}{27}\left(\sum_{cyc}x\right)^4\underset{(*)}{\overset{?}{\le}}\lambda\left(\frac{1}{27}\left(\sum_{cyc}x\right)^3-xyz\right)\left(\sum_{cyc}x\right)\left(\because\sum_{cyc}x=1\right)$$

$$\text{and}\because\lambda\ge 26\text{ and }\frac{1}{27}\left(\sum_{cyc}x\right)^3-xyz\overset{AM-GM}{\ge}0\therefore\text{ in order to prove }(*),$$

$$\text{it suffices to prove : }26\left(\left(\sum_{cyc}x\right)^3-27xyz\right)\left(\sum_{cyc}x\right)\overset{?}{\ge}27\sum_{cyc}x^4-\left(\sum_{cyc}x\right)^4$$

$$\Leftrightarrow 2\sum_{cyc}x^3y+2\sum_{cyc}xy^3+3\sum_{cyc}x^2y^2\underset{(**)}{\overset{?}{\ge}}7xyz\left(\sum_{cyc}x\right)$$

$$\text{Indeed, via AM}-\text{GM}, 2\sum_{cyc}x^3y+2\sum_{cyc}xy^3\ge 4\sum_{cyc}x^2y^2\ge 4xyz\left(\sum_{cyc}x\right)$$

$$\text{and }3\sum_{cyc}x^2y^2\ge 3xyz\left(\sum_{cyc}x\right)\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore x^4+y^4+z^4+\lambda xyz\le\frac{\lambda+1}{27}\ \forall x,y,z>0\,|\,x+y+z=1\text{ and }\lambda\ge 26,$$

$$''=''\text{ iff }x=y=z=\frac{1}{3}\text{ (QED)}$$

Solution 2 by proposer

Using pqr – Method

We denote $p=x+y+z, q=xy+yz+zx, r=xyz$.

We have $p=1, p^2\ge 3q\Rightarrow$

$$x^4+y^4+z^4=p^4-4p^2q+2q^2+4pr=1-4q+2q^2+4r.$$

Inequality $x^4+y^4+z^4+\lambda xyz\le\frac{\lambda+1}{27}$ we write: $(1-4q+2q^2+4r)+\lambda r\le\frac{\lambda+1}{27}\Leftrightarrow$

$2q^2-4q+(\lambda+4)r\le\frac{\lambda-26}{27}$, which follows from: $1\ge 3q$ and $q^2\ge 3r$.

We obtain:

$$2q^2-4q+(\lambda+4)r\le 2q^2-4q+(\lambda+4)\frac{q^2}{3}\overset{(1)}{\le}\frac{\lambda-26}{27},$$

where (1) $\Leftrightarrow 2q^2-4q+(\lambda+4)\frac{q^2}{3}\le\frac{\lambda-26}{27}\Leftrightarrow 54q^2-108q+9(\lambda+4)q^2\le\lambda-26\Leftrightarrow$

$$\Leftrightarrow(9\lambda+90)q^2-108q+26-\lambda\le 0\Leftrightarrow(3\lambda-1)[3(\lambda+10)q+\lambda-26]\le 0,$$

which follows from $1\ge 3q$ and $3(\lambda+10)q+\lambda-26>0$.

Equality holds if and only if $x=y=z=\frac{1}{3}$.

JP.592 Solve for real numbers:

$$\begin{cases}2^x+3^y+5^z=10\\ \left|\sqrt{x^2+y^2+z^2}-\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}\right|=\left|x-\frac{1}{x}\right|+\left|y-\frac{1}{y}\right|+\left|z-\frac{1}{z}\right|\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Upon squaring, }\left|\sqrt{x^2+y^2+z^2}-\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}\right|=$$

$$\left|x-\frac{1}{x}\right|+\left|y-\frac{1}{y}\right|+\left|z-\frac{1}{z}\right|\Rightarrow\sum_{cyc}x^2+\sum_{cyc}\frac{1}{x^2}-2.\sqrt{\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}\frac{1}{x^2}\right)}=$$

$$\sum_{cyc}x^2+\sum_{cyc}\frac{1}{x^2}-6+2\sum_{cyc}\left(\left|x-\frac{1}{x}\right|\left|y-\frac{1}{y}\right|\right)$$

$$\Rightarrow 3\overset{(*)}{\ge}\sqrt{\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}\frac{1}{x^2}\right)}+\sum_{cyc}\left(\left|x-\frac{1}{x}\right|\left|y-\frac{1}{y}\right|\right)\overset{(**)}{\ge}\sqrt{\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}\frac{1}{x^2}\right)}$$

$$\left(\begin{array}{c}'' ='' \text{ for } (**), \text{i.e.} \sum_{cyc}\left(\left|x-\frac{1}{x}\right|\left|y-\frac{1}{y}\right|\right)=0 \text{ occurs iff :} \\ \left|x-\frac{1}{x}\right|\left|y-\frac{1}{y}\right|=\left|y-\frac{1}{y}\right|\left|z-\frac{1}{z}\right|=\left|z-\frac{1}{z}\right|\left|x-\frac{1}{x}\right|=0 \Rightarrow \text{ iff} \\ \left(\left|x-\frac{1}{x}\right|=\left|y-\frac{1}{y}\right|=0\right) \text{ or } \left(\left|y-\frac{1}{y}\right|=\left|z-\frac{1}{z}\right|=0\right) \text{ or } \left(\left|z-\frac{1}{z}\right|=\left|x-\frac{1}{x}\right|=0\right) \\ \text{or } \left(\left|x-\frac{1}{x}\right|=\left|y-\frac{1}{y}\right|=\left|z-\frac{1}{z}\right|=0\right) \\ \Rightarrow \sum_{cyc}\left(\left|x-\frac{1}{x}\right|\left|y-\frac{1}{y}\right|\right)=0 \text{ iff } \begin{Bmatrix} x=\pm 1 \\ y=\pm 1\end{Bmatrix} \text{ or } \begin{Bmatrix} y=\pm 1 \\ z=\pm 1\end{Bmatrix} \text{ or } \begin{Bmatrix} z=\pm 1 \\ x=\pm 1\end{Bmatrix} \text{ or } \begin{Bmatrix} x=\pm 1 \\ y=\pm 1 \\ z=\pm 1\end{Bmatrix}\end{array}\right)$$

$$\Rightarrow 9 \geq \left(\sum_{cyc} x^2\right)\left(\sum_{cyc}\frac{1}{x^2}\right); \text{but via AM-HM,} \left(\sum_{cyc} x^2\right)\left(\sum_{cyc}\frac{1}{x^2}\right) \geq 9$$

$$(\because x^2, y^2, z^2 > 0 \text{ as } x, y, z \neq 0) \therefore \text{ we must have : } \left(\sum_{cyc} x^2\right)\left(\sum_{cyc}\frac{1}{x^2}\right)=9,$$

$$\text{and it's the equality case of } \left(\sum_{cyc} x^2\right)\left(\sum_{cyc}\frac{1}{x^2}\right) \geq 9 \text{ and which occurs iff :}$$

$$x^2=y^2=z^2 \text{ and again, } 3=\sqrt{\left(\sum_{cyc} x^2\right)\left(\sum_{cyc}\frac{1}{x^2}\right)} \overset{\text{via } (*)}{\Rightarrow} \sum_{cyc}\left(\left|x-\frac{1}{x}\right|\left|y-\frac{1}{y}\right|\right)=0$$

$$\text{and as stated earlier, it occurs iff : } \begin{Bmatrix} x=\pm 1 \\ y=\pm 1\end{Bmatrix} \text{ or } \begin{Bmatrix} y=\pm 1 \\ z=\pm 1\end{Bmatrix} \text{ or } \begin{Bmatrix} z=\pm 1 \\ x=\pm 1\end{Bmatrix}$$

$$\text{or } \begin{Bmatrix} x=\pm 1 \\ y=\pm 1 \\ z=\pm 1\end{Bmatrix} \text{ and combining with } x^2=y^2=z^2, \text{we get : } \begin{Bmatrix} x=\pm 1 \\ y=\pm 1 \\ z=\pm 1\end{Bmatrix}$$

$$\text{and among this final sets of values, the only set that satisfies :}$$

$$2^x+3^y+5^z=10 \text{ is : } \begin{Bmatrix} x=1 \\ y=1 \\ z=1\end{Bmatrix} \text{ (answer)}$$

Solution 2 by proposer

$$x, y, z \in \mathbb{R} \setminus \{0\}$$

Denote: $R=x^2+y^2+z^2; S=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$

$$\sqrt{R}=\sqrt{x^2+y^2+z^2}>\sqrt{x^2}=|x|$$

Analogous: $\sqrt{R}>|y|; \sqrt{R}>|z|$

$$\sqrt{S}=\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}>\sqrt{\frac{1}{x^2}}=\frac{1}{|x|};\sqrt{S}>\frac{1}{|y|};\sqrt{S}>\frac{1}{|z|}$$

$$\frac{\left|x+\frac{1}{x}\right|}{\sqrt{R}+\sqrt{S}}\le\frac{|x|+\frac{1}{|x|}}{\sqrt{R}+\sqrt{S}}<\frac{|x|+\frac{1}{|x|}}{|x|+\frac{1}{|x|}}=1 \quad (1)$$

$$\frac{\left|y+\frac{1}{y}\right|}{\sqrt{R}+\sqrt{S}}\le\frac{|y|+\frac{1}{|y|}}{\sqrt{R}+\sqrt{S}}<\frac{|y|+\frac{1}{|y|}}{|y|+\frac{1}{|y|}}=1 \quad (2)$$

$$\frac{\left|z+\frac{1}{z}\right|}{\sqrt{R}+\sqrt{S}}\le\frac{|z|+\frac{1}{|z|}}{\sqrt{R}+\sqrt{S}}<\frac{|z|+\frac{1}{|z|}}{|z|+\frac{1}{|z|}}=1 \quad (3)$$

$$\left|\sqrt{x^2+y^2+z^2}-\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}\right|=\left|\sqrt{R}-\sqrt{S}\right|=\frac{\left|\sqrt{R}-\sqrt{S}\right|}{1}=\frac{|R-S|}{\left|\sqrt{R}+\sqrt{S}\right|}=$$

$$=\frac{\left|x^2+y^2+z^2-\frac{1}{x^2}-\frac{1}{y^2}-\frac{1}{z^2}\right|}{\sqrt{R}+\sqrt{S}}=\frac{\left|\left(x^2-\frac{1}{x^2}\right)+\left(y^2-\frac{1}{y^2}\right)+\left(z^2-\frac{1}{z^2}\right)\right|}{\sqrt{R}+\sqrt{S}}=$$

$$\le\frac{\left|x^2-\frac{1}{x^2}\right|+\left|y^2-\frac{1}{y^2}\right|+\left|z^2-\frac{1}{z^2}\right|}{\sqrt{R}+\sqrt{S}}\overset{(1);(2);(3)}{\le}\left|x-\frac{1}{x}\right|+\left|y-\frac{1}{y}\right|+\left|z-\frac{1}{z}\right|$$

Equality holds for:

$$x=\frac{1}{x};y=\frac{1}{y};z=\frac{1}{z}\Rightarrow x,y,z\in\{-1,1\}$$

$$2^x+3^y+5^z=10\Longrightarrow x=y=z=1$$

JP.593 Solve for real numbers:

$$\frac{x^2}{x^2+4\sqrt{x+2}}+\frac{2}{2+x\sqrt{x+2}}=\frac{4x}{5x+2}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$(x-2)^2\ge 0\Rightarrow x^2-4x+4\ge 0\Rightarrow x^2+4\ge 4x$$

$$x+2+x^2+4\ge x+2+4x\Rightarrow 5x+2\le x^2+x+6$$

$$\frac{1}{5x+2}\ge\frac{1}{x^2+x+6}$$

$$\frac{x+2}{5x+2} \ge \frac{x+2}{x^2+x+6} \qquad (1)$$

$$\left(\sqrt{x+2}-2\right)^2 \ge 0 \Rightarrow x+2-4\sqrt{x+2}+4 \ge 0$$

$$x+6 \ge 4\sqrt{x+2} \Rightarrow x^2+x+6 \ge x^2+4\sqrt{x+2}$$

$$\frac{1}{x^2+4\sqrt{x+2}} \ge \frac{1}{x^2+x+6}$$

$$\frac{x^2}{x^2+4\sqrt{x+2}} \ge \frac{x^2}{x^2+x+6} \qquad (2)$$

$$\left(\sqrt{x+2}-x\right)^2 \ge 0 \Rightarrow x+2-2x\sqrt{x+2}+x^2 \ge 0$$

$$x^2+x+2 \ge 2x\sqrt{x+2}$$

$$x^2+x+6 \ge 4+2x\sqrt{x+2}$$

$$\frac{1}{4+2x\sqrt{x+2}} \ge \frac{1}{x^2+x+6}$$

$$\frac{4}{4+2x\sqrt{x+2}} \ge \frac{4}{x^2+x+6}$$

$$\frac{2}{2+x\sqrt{x+2}} \ge \frac{4}{x^2+x+6} \qquad (3)$$

By adding (1); (2); (3):

$$\frac{x+2}{5x+2}+\frac{x^2}{x^2+4\sqrt{x+2}}+\frac{2}{2+x\sqrt{x+2}} \ge$$

$$\ge \frac{x+2}{x^2+x+6}+\frac{x^2}{x^2+x+6}+\frac{2}{x^2+x+6} = \frac{x^2+x+6}{x^2+x+6} = 1$$

$$\frac{x+2}{5x+2}+\frac{x^2}{x^2+4\sqrt{x+2}}+\frac{2}{2+x\sqrt{x+2}} \ge 1$$

$$\frac{x^2}{x^2+4\sqrt{x+2}}+\frac{2}{2+x\sqrt{x+2}} \ge 1-\frac{x+2}{5x+2}$$

$$\frac{x^2}{x^2+4\sqrt{x+2}}+\frac{2}{2+x\sqrt{x+2}} \ge \frac{4x}{5x+2} \qquad (4)$$

Equality holds in (4) for:

$$\sqrt{x+2} = x = 2$$

Solution: $x = 2$.

JP.594 Solve for real numbers :

$$\sin^{2022} x \cdot \cos^{2024} x = \frac{1}{2^{2022}}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sin^{2022} x \cdot \cos^{2024} x = \frac{1}{2^{2022}} \Rightarrow (\sin^2 x)^{1011}(\cos^2 x)^{1011} \cdot \cos^2 x = \left(\frac{1}{4}\right)^{1011}$$

$$\Rightarrow (\sin^2 x)(\cos^2 x) \cdot (\cos^2 x)^{\frac{1}{1011}} = \frac{1}{4} \Rightarrow 4(\sin^2 x)(\cos^2 x) \cdot (\cos^2 x)^{\frac{1}{1011}} = 1$$

$$\Rightarrow 1 = (\sin^2 2x) \cdot (\cos^2 x)^{\frac{1}{1011}} \overset{\sin^2 2x \le 1}{\le} (\cos^2 x)^{\frac{1}{1011}} \left(\because (\cos^2 x)^{\frac{1}{1011}} \ge 0\right) \Rightarrow$$

$$0 \le \frac{1}{1011} \cdot \ln(\cos^2 x) \left(\begin{matrix}\because \cos^2 x \ne 0 \text{ as } \cos^2 x = 0 \text{ makes} \\ \sin^{2022} x \cdot \cos^{2024} x = 0\end{matrix}\right)$$

$$\Rightarrow \ln(\cos^2 x) \ge 0; \text{but } \because 0 < \cos^2 x \le 1, \text{hence } \ln(\cos^2 x) \le 0 \text{ and so}$$

$$\text{only possibility is : } \ln(\cos^2 x) = 0 \Rightarrow \cos^2 x = 1 \Rightarrow \sin^2 x = 0$$

$$\Rightarrow \sin^{2022} x \cdot \cos^{2024} x = 0, \text{but } \sin^{2022} x \cdot \cos^{2024} x = \frac{1}{2^{2022}}$$

$$\Rightarrow \cos^2 x \text{ cannot be } = 1 \text{ and so}$$

$$\sin^{2022} x \cdot \cos^{2024} x = \frac{1}{2^{2022}} \text{ admits no real solution (answer)}$$

JP.595 Find $x, y, z > 1$ such that:

$$\sum_{cyc} \frac{\log_2 x}{\log_2^6 x + \log_2^3 y + \log_2^3 z} = \frac{1}{27}\left(\sum_{cyc} \log_2 x\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Denote: $a = \log_2 x\,; b = \log_2 y\,; c = \log_2 z$

$$x, y, z > 1 \Rightarrow a, b, c > 0$$

$$\sum_{cyc} \frac{\log_2 x}{\log_2^6 x + \log_2^3 y + \log_2^3 z} = \sum_{cyc} \frac{a}{a^6 + b^3 + c^3} \le$$

$$\overset{AM-GM}{\leq} \sum_{cyc} \frac{a}{3\sqrt[3]{a^6 \cdot b^3 \cdot c^3}} = \frac{1}{3}\sum_{cyc} \frac{a}{a^2bc} = \frac{1}{3}\sum_{cyc} \frac{1}{abc} = \frac{1}{3} \cdot \frac{3}{abc} = \frac{1}{abc} \leq$$

$$\overset{GM-HM}{\leq} \frac{1}{\left(\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\right)^3} = \frac{1}{27} \cdot \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3 =$$

$$= \frac{1}{27}\left(\log_x 2 + \log_y 2 + \log_z 2\right)^3 = \frac{1}{27}\left(\sum_{cyc} \log_x 2\right)^3$$

Equality holds for:

$$\begin{cases} a^6 = b^3 = c^3 \\ b^6 = c^3 = a^3 \\ c^6 = a^3 = b^3 \end{cases} \Rightarrow a = b = c = 1 \Rightarrow \log_2 x = \log_2 y = \log_2 z \Rightarrow x = y = z = 2$$

JP.596 In acute $\Delta ABC, AA', BB', CC'$ - are altitudes, $C' \in (AB), B' \in (AC)$, $\{H\} = BB' \cap CC'$ and E, F are middle points of $[BH], [AC]$ respectively.

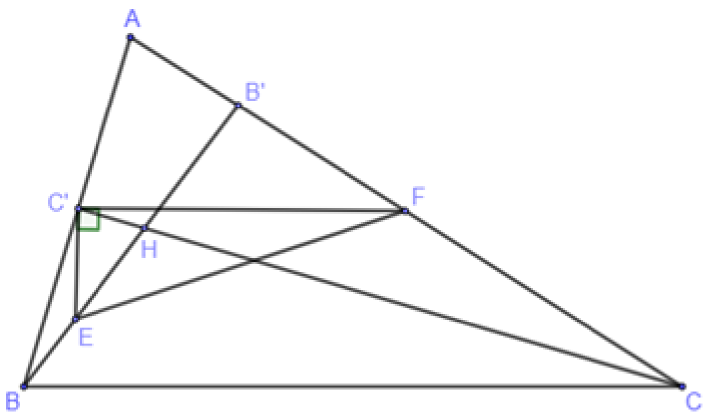
Prove that:

$$4EF^2 \geq (EC' + EB')^2 + (C'F + B'F)^2$$

Proposed by Marian Ursărescu, Florică Anastase – Romania

Solutions by proposers

We have:



$$\overrightarrow{C'F}\cdot\overrightarrow{C'E}=\frac{1}{2}\left(\overrightarrow{C'A}+\overrightarrow{C'C}\right)\cdot\frac{1}{2}\left(\overrightarrow{C'B}+\overrightarrow{C'H}\right)=$$

$$=\frac{1}{4}\left(\overrightarrow{C'A}\cdot\overrightarrow{C'B}+\overrightarrow{C'A}\cdot\overrightarrow{C'H}+\overrightarrow{C'C}\cdot\overrightarrow{C'B}+\overrightarrow{C'C}\cdot\overrightarrow{C'H}\right)=$$

$=\frac{1}{4}\left(\overrightarrow{C'A}\cdot\overrightarrow{C'B}+0+0+\overrightarrow{C'C}\cdot\overrightarrow{C'H}\right)=\frac{1}{4}\left(-\overrightarrow{C'A}\cdot\overrightarrow{C'B}+\overrightarrow{C'C}\cdot\overrightarrow{C'H}\right)=0$ because $\Delta AC'C\sim$

$$\Delta HC'B\Rightarrow\frac{BC'}{C'C}=\frac{C'H}{C'A}\Leftrightarrow\overrightarrow{C'A}\cdot\overrightarrow{C'B}=\overrightarrow{C'C}\cdot\overrightarrow{C'H}$$

Hence, $\overrightarrow{C'F}\cdot\overrightarrow{C'E}=0\Leftrightarrow C'F\perp C'E$. So, we have:

$EF^2=(EC')^2+(C'F)^2$ and $EF^2=(EB')^2+(B'F)^2$

$$2EF=\sqrt{(EC')^2+(C'F)^2}+\sqrt{(EB')^2+(B'F)^2}\geq\sqrt{(EC'+EB')^2+(C'F+B'F)^2},$$

therefore we obtain $4EF^2\geq(EC'+EB')^2+(C'F+B'F)^2$

JP.597 Let $ABCD$ be an convex quadrilateral, $\lambda\in\mathbb{R}$ and M,N be such that $\overrightarrow{AN}=\lambda\cdot\overrightarrow{AB},\overrightarrow{DN}=\lambda\cdot\overrightarrow{DC},\overrightarrow{AD}=3\cdot\overrightarrow{BC}$. Find $\lambda\in\mathbb{R}$ such that $\overrightarrow{MN}=7\cdot\overrightarrow{BC}$

Proposed by Marian Ursărescu, Florică Anastase – Romania

Solution by proposers

We have:

$\overrightarrow{MN}=\overrightarrow{MB}+\overrightarrow{BC}+\overrightarrow{CN}\Rightarrow\lambda\cdot\overrightarrow{MN}=\lambda\left(\overrightarrow{MB}+\overrightarrow{BC}+\overrightarrow{CN}\right)$ and

$(1-\lambda)\overrightarrow{MN}=(1-\lambda)\left(\overrightarrow{MA}+\overrightarrow{AD}+\overrightarrow{DN}\right)$, hence

$$\overrightarrow{MN}=(\lambda+1-\lambda)\overrightarrow{MN}=\lambda\left(\overrightarrow{MB}+\overrightarrow{BC}+\overrightarrow{CN}\right)+(1-\lambda)\left(\overrightarrow{MA}+\overrightarrow{AD}+\overrightarrow{DN}\right)=$$

$$=\lambda\overrightarrow{BC}+(1-\lambda)\overrightarrow{AD}+\left[\lambda\overrightarrow{MB}+(1-\lambda)\overrightarrow{MA}\right]+\left[\lambda\overrightarrow{CN}+(1-\lambda)\overrightarrow{DN}\right]$$

Let be $CE\parallel AB, CE=AB$. From $3\overrightarrow{BC}=\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$, it follows $2\overrightarrow{BC}=\overrightarrow{AB}+\overrightarrow{CD}=\overrightarrow{EC}+\overrightarrow{CD}=\overrightarrow{ED}$, then $DE\parallel BE, DE=2BC$, and hence, $ABCD$ is an parallelogram, so $AE\parallel BC, AE=BC$.

Therefore, $\overrightarrow{MN}=\lambda\overrightarrow{BC}+(1-\lambda)\overrightarrow{AD}$ and $\overrightarrow{MN}=(3-2\lambda)\overrightarrow{BC}$, thus

$$\overrightarrow{MN}=7\overrightarrow{BC}\Leftrightarrow 3-2\lambda=7\Leftrightarrow\lambda=-2.$$

JP.598 Let $n \geq 4$, and let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$. Prove that:

$$\frac{1}{2a_1+5} + \frac{1}{2a_2+5} + \cdots + \frac{1}{2a_n+5} \geq \frac{n}{7}$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Denoting

$$x = \frac{a_2 + \cdots + a_{n-1}}{n-2}, \quad S = \frac{a_1 + a_n}{2}$$

by Lemma below we have

$$(n-3)x^2 + 2Sx + a_1a_n \leq n.$$

By the AM-HM inequality,

$$\frac{1}{2a_2+5} + \cdots + \frac{1}{2a_{n-1}+5} \geq \frac{(n-2)^2}{(2a_2+5) + \cdots + (2a_{n-1}+5)} = \frac{n-2}{2x+5}$$

So, it suffices to show that

$$\frac{1}{2a_1+5} + \frac{1}{2a_n+5} + \frac{n-2}{2x+5} \geq \frac{n}{7}$$

for $a_1 \geq x \geq a_n \geq 0$ and $(n-3)x^2 + 2Sx + a_1a_n \leq n$. Since the left side of the required inequality decreases if each of a_1, a_n and x increases, it is enough to consider the case

$$(n-3)x^2 + 2Sx + a_1a_n = n.$$

Since $2S = a_1 + a_2 \geq a_1 \geq x$, we get

$$n = (n-3)x^2 + 2Sx + a_1a_n \geq (n-3)x^2 + x^2 + a_1a_n \geq (n-2)x^2,$$

hence

$$0 < x \leq x_1, x_1 = \sqrt{\frac{n}{n-2}}.$$

From $(x-a_1)(x-a_n) \leq 0$, we get $a_1a_n \leq 2Sx - x^2$, hence

$$n = (n-3)x^2 + 2Sx + a_1a_n \leq (n-3)x^2 + 2Sx + (2Sx - x^2) \leq (n-4)x^2 + 4Sx,$$

$$S \geq S_1 = \frac{n-(n-4)x^2}{4x}$$

On the other hand, since

$$\frac{1}{2a_1+5}+\frac{1}{2a_n+5}=\frac{4S+10}{4a_1a_n+20S+25}=\frac{4S+10}{4n+25-4(n-3)x^2+4(5-2x)S},$$

the required inequality becomes

$$\frac{2S+5}{4n+25-4(n-3)x^2+4(5-2x)S}+\frac{n-7-nx}{7(2x+5)}\geq 0,$$

which can be written as

$$2A(x)S+B(x)\geq 0$$

where

$$A(x)=4nx^2-14(n-3)x+5(2n-7)=4n\left(x-\frac{7n-21}{4n}\right)^2+\frac{-9n^2+154n-441}{4n},$$

$$B(x)=4n(n-3)x^3-4(n-3)(n-7)x^2-(4n^2+25n-70)x+n(4n-3).$$

We claim that $A(x)\geq 0$. For $n\in\{4,5,\dots,13\}$, we have

$$A(x)\geq\frac{-9n^2+154n-441}{4n}>\frac{-9n^2+151n-442}{4n}=\frac{(13-n)(9n-34)}{4n}\geq 0.$$

For $n\geq 13$, since

$$x\leq x_1=\sqrt{1+\frac{2}{n-2}}<1+\frac{1}{n-2}=\frac{n-1}{n-2}<\frac{7n-21}{4n},$$

$A(x)$ is decreasing, hence

$$A(x)\geq A(x_1)=\frac{4n^2}{n-2}-14(n-3)x_1+5(2n-7)=\frac{14n^2-55n+70}{n-2}-14(n-3)x_1$$

$$>\frac{14(n-3)(n-1)}{n-2}-14(n-3)x_1=\frac{14(n-3)\left(n-1-\sqrt{n^2-2n}\right)}{n-2}>0.$$

Since $A(x)>0$, it suffices to show that $2A(x)S_1+B(x)\geq 0$. This inequality is equivalent to

$$4(n-2)x^4+6(n-3)x^3-(14n-25)x^2-6(n-6)x+5(2n-7)\geq 0,$$

$$(x-1)^2[4(n-2)x^2+2(7n-17x)+5(2n-7)]\geq 0$$

Clearly, the last inequality is true for $n\geq 4$. So, the proof is completed. The equality occurs for $a_1=a_2=\cdots=a_n=1$.

Lemma. If $n\geq 4$ and $a_1,a_2,\dots,a_n$ are nonnegative real numbers such that

$a_1 \geq a_2 \geq \cdots . \geq a_n$ and $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$, then:

$$(n-3)x^2 + (a_1 + a_n)x + a_1a_n \leq n,$$

where $x = \frac{a_2 + \cdots + a_{n-1}}{n-2}$.

Proof. Write the desired inequality in the homogeneous form:

$$(n-3)x^2 + (a_1 + a_n)x + a_1a_n \leq a_1a_2 + a_2a_3 + \cdots + a_na_1,$$

which is equivalent to

$$(n-3)x^2 + a_1(x - a_2) + a_n(x - a_{n-1}) \leq a_2a_3 + \cdots + a_{n-2}a_{n-1}.$$

Since $x - a_1 \leq 0$ and $x - a_{n-1} \geq 0$, it suffices to show that

$$(n-3)x^2 + a_2(x - a_2) + a_{n-1}(x - a_{n-1}) \leq a_2a_3 + \cdots + a_{n-2}a_{n-1},$$

which can be rewritten as

$$a_2a_3 + \cdots + a_{n-2}a_{n-1} \geq (n-3)x^2 + (a_2 + a_{n-1})x - a_2^2 - a_{n-1}^2.$$

Since the sequences $a_2, a_3, \dots, a_{n-2}$ and $a_3, a_4, \dots, a_{n-1}$ are decreasing, by Chebyshev's inequality we have

$$(n-3)(a_2a_3 + \cdots + a_{n-2}a_{n-1}) \geq (a_2 + \cdots + a_{n-2})(a_3 + \cdots + a_{n-1}) =$$

$= \big((n-2)x - a_{n-1}\big)\big((n-2)x - a_2\big)$. Thus, it suffices to show that

$$\frac{\big((n-2)x - a_{n-1}\big)\big((n-2)x - a_2\big)}{n-3} \geq (n-3)x^2 + (a_2 + a_{n-1})x - a_2^2 - a_{n-1}^2,$$

which is equivalent to

$$(2n-5)x^2 - (2n-5)(a_2 + a_{n-1})x + (n-3)\big(a_2^2 + a_{n-1}^2\big) + a_2a_{n-1} \geq 0,$$

$$(2n-5)(2x - a_2 - a_{n-1})^2 + (2n-7)(a_2 - a_{n-1})^2 \geq 0.$$

Clearly, the latter inequality is true.

JP.599 Prove that 3 is the largest positive value of the power k such that the inequality: $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq a_1^k + a_2^k + \cdots + a_n^k$

holds for $n \geq 2$ and any positive real numbers $a_1, a_2, \dots, a_n$ with at most one $a_i < 1$ and $a_1^2 + a_2^2 + \cdots + a_n^2 = n$.

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

For $a_1 = (1+t)^{\frac{1}{2}}, a_2 = \cdots = a_{n-1} = 1$ and $a_n = (1-t)^{\frac{1}{2}}$, where $t \in [0,1)$, the constraints are satisfied, and the inequality is equivalent to $E(t) \geq 0$, where

$$E(t) = (1+t)^{-\frac{1}{2}} + (1-t)^{-\frac{1}{2}} + (1+t)^{\frac{k}{2}} - (1-t)^{\frac{k}{2}}.$$

We have

$$2E'(t) = -(1+t)^{-\frac{3}{2}} + (1-t)^{-\frac{3}{2}} - k(1+t)^{\frac{k}{2}-1} + k(1-t)^{\frac{k}{2}-1},$$

$$4E''(t) = 3(1+t)^{-\frac{5}{2}} + 3(1-t)^{-\frac{5}{2}} - k(k-2)(1+t)^{\frac{k}{2}-2} - k(k-2)(1-t)^{\frac{k}{2}-2}$$

Since $E(0) = E'(0) = 0$, the condition $E''(0) \geq 0$ is necessary to have $E(t) \geq 0$ for $t \in [0,1)$.

From

$$4E''(0) = 2(3-k)(1+k) \geq 0,$$

we get $k \leq 3$. To show that 3 is the largest positive value of k, we need to prove that $F \geq 0$ for $n \geq 2$ and all positive a_i satisfying $a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n$ and $a_1^2 + a_2^2 + \cdots + a_n^2 = n$,

where

$$F = \frac{1}{a_1} + \frac{1}{a_2} + \cdots . + \frac{1}{a_n} - a_1^3 - a_2^3 - \cdots - a_n^3.$$

We use the induction method. For $n = 2$, we have

$$F = (a_1 + a_2)\left(\frac{1}{a_1 a_2} - a_1^2 + a_1 a_2 - a_2^2\right) = \frac{(a_1 + a_2)(a_1 a_2 - 1)^2}{a_1 a_2} \geq 0.$$

For $n \geq 3$, assume that a_1 and $a_3, \dots, a_{n-1}$ are fixed, and a_n and E are functions of a_2.

We have $a_n a_n' + a_2 = 0$ and

$$F'(a_2) = \left(-\frac{1}{a_n^2} - 3a_n^2\right)a_n' - \frac{1}{a_2^2} - 3a_2^2 = \left(\frac{1}{a_n^2} + 3a_n^2\right)\frac{a_2}{a_n} - \frac{1}{a_2^2} - 3a_2^2$$

$$= 3a_2(a_n - a_2) + \frac{a_2^3 - a_n^3}{a_2^2 a_n^3} = \frac{(a_n - a_2)(3a_2^3 a_n^3 - a_2^2 - a_2 a_n - a_n^2)}{a_2^2 a_n^3}$$

$$\geq \frac{(a_n - a_2)(3a_n^2 - a_2^2 - a_2 a_n - a_n^2)}{a_2^2 a_n^3} = \frac{(a_n - a_2)^2(2a_n + a_2)}{a_2^2 a_n^3} \geq 0.$$

From $F'(a_2) \geq 0$, it follows that $F(a_2)$ is increasing and has the minimum value when a_2 is minimum, hence when $a_2 = 1$. So, it suffices to consider this case, when the inequality holds from the induction hypothesis. The proof is finished. If $k = 3$, then the equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

JP.600 **Calculate the limit of sequence $(a_n)_{n\geq 1}$ defined by the following relationship:**

$$a_n = \frac{1}{n}\int_0^{\frac{1}{2}} \ln\left(1 + e^{n\cdot \arcsin x}\right) dx$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by Angel Plaza – Spain

For any $u \geq 0$, the inequality $u \leq \ln(1 + e^u) \leq (2e^u) = u + \ln 2$ holds. By doing $u = n \cdot \arcsin x$, it is obtained

$$\arcsin x \leq \frac{\ln(1 + e^{n\cdot \arcsin x})}{n} \leq n \cdot \arcsin x + \frac{\ln 2}{n}.$$

By integrating the inequality over the interval $\left[0, \frac{1}{2}\right]$, we get:

$$\int_0^{\frac{1}{2}} \arcsin x\, dx \leq a_n \leq \int_0^{\frac{1}{2}} \arcsin x\, dx + \frac{\ln 2}{2n}.$$

As $n \to \infty$, the term $\frac{\ln 2}{2n}$ approaches 0. Then, by the Squeeze theorem, the limit is

$$\lim_{n\to\infty} a_n = \int_0^{\frac{1}{2}} \arcsin x\, dx = x \arcsin x + \sqrt{1 - x^2}\Big]_0^{\frac{1}{2}} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

Solution 2 by proposer

We use the double inequality, which is easy to demonstrate:

$$e^{n\cdot \arcsin x} < 1 + e^{n\cdot \arcsin x} < 2 \cdot e^{n\cdot \arcsin x},$$

for $x > 0$ and $n \in \mathbb{N}^*$.

Through logarithm, we obtain:

$$n \cdot \arcsin x < \ln(1 + e^{n\cdot \arcsin x}) < \ln 2 + n \cdot \arcsin x.$$

We integrate over the interval $\left[1,\frac{1}{2}\right]$.

We use the following relationship which we obtain integrating by parts:

$$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + C$$

We have:

$$n\cdot\left(\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1\right) < \int_0^{\frac{1}{2}} \ln\left(1+e^{n\cdot\arcsin x}\right)dx < \frac{\ln 2}{2} + n\cdot\left(\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1\right)$$

We divide by n:

$$\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1 < \frac{1}{n}\int_0^{\frac{1}{2}} \ln\left(1+e^{n\cdot\arcsin x}\right)dx < \frac{1}{n}\cdot\frac{\ln 2}{2}+\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1$$

In this double inequality let's go to the limit for $n\to\infty$.

According to the squeeze theorem we obtain the limit required in the statement of the problem, which has the value:

$$a = \frac{\pi}{12}+\frac{\sqrt{3}}{2}-1.$$

So, this problem is solved.

PROBLEMS FOR SENIORS

SP.586 Solve for real numbers:

$$\begin{cases}\left(\sqrt{x+y}-\sqrt{x}\right)\left(\sqrt{x^2+xy}+1\right)=xy\\ x+y+z=3\\ \left(\sqrt{y+z}-\sqrt{y}\right)\left(\sqrt{y^2+yz}+1\right)=xy\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{cases}\left(\sqrt{x+y}-\sqrt{x}\right)\left(\sqrt{x^2+xy}+1\right)=xy \to (1)\\ x+y+z=3\to (2)\\ \left(\sqrt{y+z}-\sqrt{y}\right)\left(\sqrt{y^2+yz}+1\right)=xy\to (3)\end{cases}$$

Since $x,y,z\in\mathbb{R}$, (1), (2) and (3) $\Rightarrow x,y\geq 0$

If $x=0$, then : (1) $\Rightarrow y=0 \overset{\text{via }(2)}{\Rightarrow} z=3$, but $\begin{Bmatrix}x=0\\y=0\\z=3\end{Bmatrix}$ doesn't satisfy (3) $\therefore x\neq 0$

$$(1)\Rightarrow\left(\frac{x+y-x}{\sqrt{x+y}+\sqrt{x}}\right)\left(\sqrt{x^2+xy}+1\right)\overset{(*)}{=}xy\left(\begin{matrix}\because x\neq 0\wedge x,y\geq 0\Rightarrow\\ \sqrt{x+y}+\sqrt{x}>0\end{matrix}\right)$$

$(*)\Rightarrow$ one possibility is $y=0\overset{\text{via }(3)}{\Rightarrow} z=0\overset{\text{via }(2)}{\Rightarrow} x=3$ and

$\begin{Bmatrix}x=3\\y=0\\z=0\end{Bmatrix}$ indeed satisfies (1), (2) and (3) and now, if $y\neq 0$, $(*)\overset{\text{via squaring}}{\Rightarrow}$

$$x^2+xy+1+2.\sqrt{x^2+xy}=x^2\left(2x+y+2.\sqrt{x^2+xy}\right)\Rightarrow$$

$$x^2(1-x)+(1-x)(1+x+x^2)+xy(1-x)+2.\sqrt{x^2+xy}.(1-x)(1+x)=0$$

$$\Rightarrow(1-x)\left(1+x+2x^2+xy+2.\sqrt{x^2+xy}.(1+x)\right)=0\Rightarrow x=1$$

$$\left(\because x,y\neq 0\wedge x,y\geq 0\Rightarrow 1+x+2x^2+xy+2.\sqrt{x^2+xy}.(1+x)>0\right)$$

$$\overset{\text{via }(2)}{\Rightarrow} y+z=2\overset{\text{via }(3)}{\Rightarrow}\left(\sqrt{2}-\sqrt{y}\right)\left(\sqrt{2y}+1\right)=y$$

$$\Rightarrow\left(2+y-2.\sqrt{2y}\right)\left(2y+1+2.\sqrt{2y}\right)=y^2$$

$$\Rightarrow 2y^2+5y+2-y^2-8y+2.\sqrt{2y}.(2+y-2y-1)=0$$

$$\Rightarrow(y-1)(y-2)-2.\sqrt{2y}.(y-1)=0\Rightarrow(y-1)\left(y-2-2.\sqrt{2y}\right)\overset{(**)}{=}0$$

$\Rightarrow$ one possibility is $y=1\overset{\text{via }(2)}{\Rightarrow} z=1$ and $\begin{Bmatrix}x=1\\y=1\\z=1\end{Bmatrix}$ indeed satisfies (1), (2) and (3)

Now, if $y\neq 1$, $(**)\Rightarrow y=2+2.\sqrt{2y}>2\left(\because y\neq 0\wedge y\geq 0\Rightarrow 2.\sqrt{2y}>0\right)$

$$\overset{y+z=2}{\Rightarrow} z<0\overset{\substack{\text{via }(3)\\ \text{and}\\ \because x=1}}{\Rightarrow} y<\left(\sqrt{y+0}-\sqrt{y}\right)\left(\sqrt{y^2+yz}+1\right)=0\Rightarrow y<0;$$

but $y\neq 0\wedge y\geq 0\Rightarrow y>0$ *in this case* consideration $\therefore(**)\Rightarrow$

only one possibility : $y=1\therefore$ following consideration of *all* possible cases,

$\begin{Bmatrix}x=3\\y=0\\z=0\end{Bmatrix}$ *and* $\begin{Bmatrix}x=1\\y=1\\z=1\end{Bmatrix}$ are the only 2 sets of real values satisfying (1), (2) *and* (3) (*answer*)

SP.587 Let a, b, c be sides in ΔABC. If $\tan B = 2; \tan C = 3$ then:

$$a^2 + b^2 + c^2 > \frac{2F}{3}\left(3\sqrt{2} + 3\sqrt{5} + 2\sqrt{10} - 11\right)$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

It is known that: $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\tan A + 2 + 3 = \tan A \cdot 2 \cdot 3$$

$$5 + \tan A = 6 \tan A \Rightarrow \tan A = 1 \Rightarrow \widehat{A} = 45°$$

$$\tan B = \tan\left(2 \cdot \frac{B}{2}\right) = \frac{2\tan\frac{B}{2}}{1 - \tan^2\frac{B}{2}}$$

$$2 = \frac{2\tan\frac{B}{2}}{1 - \tan^2\frac{B}{2}} \Rightarrow 2\tan\frac{B}{2} = 2 - 2\tan^2\frac{B}{2}, \qquad \tan^2\frac{B}{2} + \tan\frac{B}{2} - 1 = 0$$

$$\tan\frac{B}{2} = \frac{-1+\sqrt{5}}{2} \qquad (1)$$

$$\tan C = \tan\left(2 \cdot \frac{C}{2}\right) = \frac{2\tan\frac{C}{2}}{1 - \tan^2\frac{C}{2}}$$

$$3 = \frac{2\tan\frac{C}{2}}{1 - \tan^2\frac{C}{2}} \Rightarrow 3 - 3\tan^2\frac{C}{2} = 2\tan\frac{C}{2}, \qquad 3\tan^2\frac{C}{2} + 2\tan\frac{C}{2} - 3 = 0$$

$$\tan\frac{C}{2} = \frac{-2 + \sqrt{4 + 36}}{2 \cdot 3} = \frac{-2 + 2\sqrt{10}}{2 \cdot 3} = \frac{\sqrt{10} - 1}{3}$$

$$\tan\frac{C}{2} = \frac{\sqrt{10}-1}{3} \qquad (2)$$

$$\tan A = \tan\left(2 \cdot \frac{A}{2}\right) = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}$$

$$1 = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}} \Rightarrow 1 - \tan^2\frac{A}{2} = 2\tan\frac{A}{2}$$

$$\tan^2\frac{A}{2}+2\tan\frac{A}{2}-1=0$$

$$\tan\frac{A}{2}=\frac{-2+\sqrt{8}}{2}=\frac{-2+2\sqrt{2}}{2}=\sqrt{2}-1$$

$$\tan\frac{A}{2}=\sqrt{2}-1 \qquad (3)$$

$$\frac{a^2}{4F}=\frac{a^2}{2bc\sin A}=\frac{4R^2\sin^2 A}{2\cdot 2R\sin B\cdot 2R\sin C\cdot \sin A}=$$

$$=\frac{\sin A}{\cos(B-C)+\cos A}>\frac{\sin A}{1+\cos A}=\tan\frac{A}{2}\overset{(3)}{=}\sqrt{2}-1$$

$$\frac{a^2}{4F}>\sqrt{2}-1 \qquad (4)$$

$$\frac{b^2}{4F}=\frac{b^2}{2ac\sin B}=\frac{4R^2\sin^2 B}{2\cdot 2R\sin A\cdot 2R\sin C\cdot \sin B}=$$

$$=\frac{\sin B}{2\sin A\sin C}=\frac{\sin B}{2\cdot\frac{1}{2}[\cos(A-C)-\cos(A+C)]}=$$

$$=\frac{\sin B}{\cos(A-C)-\cos(\pi-B)}=\frac{\sin B}{\cos(A-C)+\cos B}>$$

$$>\frac{\sin B}{1+\cos B}=\tan\frac{B}{2}\overset{(1)}{=}\frac{\sqrt{5}-1}{2}$$

$$\frac{b^2}{4F}>\frac{\sqrt{5}-1}{2} \qquad (5)$$

$$\frac{c^2}{4F}=\frac{c^2}{2ab\sin C}=\frac{(2R\sin C)^2}{2\cdot 2R\sin A\cdot 2R\sin B\cdot \sin C}=$$

$$=\frac{4R^2\sin^2 C}{2\cdot 4R^2\sin A\sin B\sin C}=\frac{\sin C}{2\sin A\sin B}=$$

$$=\frac{\sin C}{2\cdot\frac{1}{2}[\cos(A-B)-\cos(A+B)]}=\frac{\sin C}{\cos(A-B)-\cos(\pi-C)}=$$

$$=\frac{\sin C}{\cos(A-B)+\cos C}>\frac{\sin C}{1+\cos C}=\tan\frac{C}{2}\overset{(2)}{=}\frac{\sqrt{10}-1}{3}$$

$$\frac{c^2}{4F}>\frac{\sqrt{10}-1}{3} \qquad (6)$$

By adding (4); (5); (6):

$$\frac{a^2}{4F}+\frac{b^2}{4F}+\frac{c^2}{4F}>\sqrt{2}-1+\frac{\sqrt{5}-1}{2}+\frac{\sqrt{10}-1}{3}$$

$$\frac{a^2+b^2+c^2}{F}>4\sqrt{2}-4+2\sqrt{5}-2+\frac{4\sqrt{10}}{3}-\frac{4}{3}$$

$$a^2+b^2+c^2>\frac{F}{3}\left(6\sqrt{2}-12+6\sqrt{5}-6+4\sqrt{10}-4\right)$$

$$a^2+b^2+c^2>\frac{F}{3}\left(6\sqrt{2}+6\sqrt{5}+4\sqrt{10}-22\right)$$

$$a^2+b^2+c^2>\frac{2F}{3}\left(3\sqrt{2}+3\sqrt{5}+2\sqrt{10}-11\right)$$

SP.588 **For given $n\geq 3$, prove that 2 is the least positive value of k such that:**

$$\frac{1}{ka_1+1}+\frac{1}{ka_2+1}+\cdots+\frac{1}{ka_n+1}\geq\frac{n}{k+1}$$

for any positive real numbers a_i with at most two $a_i>1$ and $a_1a_2\ldots a_n=1$.

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

For $a_1=a_2:=x\geq 1, a_3=\cdots=a_{n-1}=1$ and $a_n=\frac{1}{x^2}$, the constraints are satisfied and the inequality becomes:

$$\frac{2}{kx+1}+\frac{x^2}{x^2+k}\geq\frac{3}{k+1}.$$

For $x\to\infty$, we get the necessary condition $1\geq\frac{3}{k+1}$, that is $k\geq 2$. To show that 2 is the least positive value of k, we will prove that $E\geq 0$ for $n\geq 3$, where

$$E=\frac{1}{2a_1+1}+\frac{1}{2a_2+1}+\cdots+\frac{1}{2a_n+1}-\frac{n}{3}.$$

We will use the induction method. For $n=2$, we have $a_1a_2=1$ and $a_1+a_2\geq 2\sqrt{a_1a_2}=2$

therefore

$$E=\frac{1}{2a_1+1}+\frac{1}{2a_2+1}-\frac{2}{3}=\frac{2(a_1+a_2+1)}{2(a_1+a_2)+5}-\frac{2}{3}=\frac{2(a_1+a_2-2)}{3(2a_1+2a_2+5)}\geq 0.$$

Assume now $n \geq 3$ and $a_1 \geq a_2 \geq \cdots \geq a_n$.

Case 1: $a_1 \geq a_2 \geq 1 \geq a_3 \geq \cdots \geq a_n$. For fixed a_i with the exception of a_2 and a_3, we may assume that a_2 is decreasing functions of a_3. Since $a_2' = -\frac{a_2}{a_3}$, we have

$$E'(a_3) = \frac{-2a_2'}{(2a_2+1)^2} - \frac{2}{(2a_3+1)^2} = \frac{2a_2}{a_3(2a_2+1)^2} - \frac{2}{(2a_3+1)^2} =$$

$$= \frac{2(a_2-a_3)(1-4a_2a_3)}{a_3(2a_2+1)^2(2a_3+1)^2}$$

If $4a_2a_3 \geq 1$, then $E'(a_3) \leq 0, E(a_3)$ is decreasing and has the minimum value when a_3 is maximum (a_2 is minimum), hence when $a_3 = 1$ or $a_2 = 1$. In both cases, the required inequality is true by the induction hypothesis. If $4a_2a_3 < 1$, then $a_3 < \frac{1}{4a_2} \leq \frac{1}{4}$ and

$$\frac{1}{2a_2+1} + \frac{1}{2a_3+1} = \frac{2(a_2+a_3+1)}{4a_2a_3+2(a_2+a_3)+1} > \frac{2(a_2+a_3+1)}{1+2(a_2+a_3)+1} = 1,$$

therefore

$$E > 1 + \frac{1}{2a_4+1} + \cdots + \frac{1}{2a_n+1} - \frac{n}{3} \geq 1 + \frac{n-3}{2a_3+1} - \frac{n}{3} > 1 + \frac{2(n-3)}{3} - \frac{n}{3} = \frac{n-3}{3}$$
$$\geq 0.$$

Case 2: $a_1 \geq 1 \geq a_2 \geq a_3 \geq \cdots \geq a_n$. For fixed a_i with the exception of a_1 and a_2, we may assume that a_1 and E are functions of a_2. Since $a_1' = -\frac{a_1}{a_2}$, we have

$$E'(a_2) = \frac{2(a_1-a_2)(1-4a_1a_2)}{a_2(2a_1+1)^2(2a_2+1)^2}.$$

If $4a_1a_2 \geq 1$, then $E'(a_2) \leq 0, E(a_2)$ is decreasing and has the minimum value when a_2 is maximum, hence when $a_2 = 1$. So, it suffices to consider the case $a_2 = 1$, when the required inequality is true by the induction hypothesis. If $4a_1a_2 < 1$, then $a_2 < \frac{1}{4a_1} \leq \frac{1}{4}$,

therefore

$$E > \frac{1}{2a_2+1} + \frac{1}{2a_3+1} + \cdots + \frac{1}{2a_n+1} - \frac{n}{3} \geq \frac{n-1}{2a_2+1} - \frac{n}{3} >$$

$$> \frac{2(n-1)}{3} - \frac{n}{3} = \frac{n-2}{3} > 0.$$

So, the proof is completed. For $k = 2$, the equality occurs when $a_1 = a_2 = \cdots = a_n = 1$.

SP.590 Solve the following system in integers $(x,y,z)\in\mathbb{N}^*\times\mathbb{N}^*\times\mathbb{Z}$

$$\begin{cases} x^3-y^2+2z=0 \\ x^2+y^2+z^2=179 \end{cases}$$

Proposed by Said Attaoui – Oran – Algeria

Solution by proposer

From the first equation, we isolate z:

$$x^3-y^2+2z=0\Rightarrow 2z=y^2-x^3\Rightarrow z=\frac{y^2-x^3}{2}$$

Substituting into the second equation, we have

$$x^2+y^2+\left(\frac{y^2-x^3}{2}\right)^2=179\Rightarrow x^2+y^2+\frac{(y^2-x^3)^2}{4}=179\Rightarrow$$

$$\Rightarrow 4x^2+4y^2+(y^2-x^3)^2=716$$

Instead of solving this algebraically, we test small natural values for x. Try $x=5\Rightarrow x^3=125$. Then we require $y^2=2z+125$ to be a perfect square. Trying $z=-13\Rightarrow y^2=99$, which is not a square. Try instead $x=3\Rightarrow x^3=27$. Take $z=11$, then, we have

$$y^2=2z+x^3=2\cdot 11+27=49\Rightarrow y=7$$

Check the second equation

$$x^2+y^2+z^2=9+49+121=179$$

So we find one solution: $(x,y,z)=(3,7,11)$.

Now notice that choosing $z=-13$ gives

$$x^3-y^2+2z=0\Rightarrow x^3=y^2-2z=y^2+26$$

We want y^2+26 to be a cube. Try $y=1\Rightarrow x^3=27\Rightarrow x=3$. So

$$x=3,y=1,z=-13$$

Check both equations, we have

$$x^3-y^2+2z=27-1-26=0,\qquad x^2+y^2+z^2=9+1+169=179$$

This gives a second valid solution. Finally,

$$(x,y,z)=(3,1,-13)\quad\text{and}\quad(3,7,11)$$

SP.591 If $a,b,c>0, a^8+b^8+c^8\le 768$ then:

$$\sum\frac{1}{\sqrt{4+a^5}}\ge\frac{1}{2}$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Using Jensen's inequality for the convex function $f:(0,\infty)\to\mathbb{R}, f(x)=\frac{1}{\sqrt{4+x}}$ we obtain:

$$LHS=\sum\frac{1}{\sqrt{4+a^5}}\overset{Jensen}{\ge}3\cdot\frac{1}{\sqrt{4+\frac{a^5+b^5+c^5}{3}}}\overset{(1)}{\ge}\frac{3}{\sqrt{4+\frac{96}{3}}}=\frac{3}{\sqrt{36}}=\frac{3}{6}=\frac{1}{2}=RHS,$$

where (1) $\Leftrightarrow a^5+b^5+c^5\le 96$, which follows from:

$$a^5+b^5+c^5=\sum a\cdot a^4\overset{CBS}{\le}\sqrt{\sum a^2\sum a^8}\overset{(2)}{\le}\sqrt{12\cdot 256}=96,$$

where (2) $\Leftrightarrow a^2+b^2+c^2\le 12$, which follows from:

$$\left(\sum a^2\right)^4\overset{Holder}{\le}27\sum a^8=27\cdot 768=20736=12^4\Rightarrow\sum a^2\le 12$$

Equality holds if and only if $a=b=c=2$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Power}-\text{Mean Inequality}\Rightarrow\left(\frac{\sum_{cyc}a^8}{3}\right)^{\frac{1}{8}}\ge\left(\frac{\sum_{cyc}a^5}{3}\right)^{\frac{1}{5}}\text{ and }\because 768\ge\sum_{cyc}a^8$$

$$\therefore\left(\frac{768}{3}\right)^{\frac{1}{8}}\ge\left(\frac{\sum_{cyc}a^5}{3}\right)^{\frac{1}{5}}\Rightarrow\sum_{cyc}a^5\le 96\to (1)\text{ and now, }\sum_{cyc}\frac{1}{\sqrt{4+a^5}}$$

$$\overset{AM-GM}{\ge}\frac{3}{\sqrt[6]{\prod_{cyc}(4+a^5)}}\overset{AM-GM}{\ge}\frac{3}{\sqrt{\frac{\sum_{cyc}(4+a^5)}{3}}}=\frac{3}{\sqrt{4+\frac{1}{3}\cdot\sum_{cyc}a^5}}\overset{via\ (1)}{\ge}\frac{3}{\sqrt{4+32}}=\frac{1}{2}$$

$$\therefore\sum_{cyc}\frac{1}{\sqrt{4+a^5}}\ge\frac{1}{2}\ \forall\, a,b,c>0\,\middle|\,a^8+b^8+c^8\le 768,$$

$$''=''\ \text{ iff } a=b=c=2\ (\text{QED})$$

SP.592 If $a,b,c>0$ and $n\in\mathbb{N}, n\ge 2$ then:

$$\sum_{cyc} a\sqrt[n]{a^n+c^n}\ge\sqrt[n]{2}(ab+bc+ca)$$

Proposed by Marin Chirciu – Romania

Solution 1 by Jose Luis Diaz – Barrero – Spain

We first use a standard inequality: for all $x,y>0$ and integers $n\ge 1$,

$$\frac{x^n+y^n}{2}\ge\left(\frac{x+y}{2}\right)^n.$$

This follows from the convexity of the function $f(t)=t^n$, as it is well-known. Multiplying both sides by 2 gives $x^n+y^n\ge 2^{1-n}(x+y)^n$ and taking the n – th root, we obtain

$$\sqrt[n]{x^n+y^n}\ge 2^{\frac{1}{n}-1}(x+y).$$

Applying this inequality to the pairs $(a,c),(b,a)$, and (c,b), we get

$$\sqrt[n]{a^n+c^n}\ge 2^{\frac{1}{n}-1}(a+c),\qquad \sqrt[n]{b^n+a^n}\ge 2^{\frac{1}{n}-1}(b+a),\qquad \sqrt[n]{c^n+b^n}\ge 2^{\frac{1}{n}-1}(c+b).$$

Multiplying each inequality by a,b and c respectively and summing cyclically, we obtain

$$\sum_{cyc} a\sqrt[n]{a^n+c^n}\ge 2^{\frac{1}{n}-1}\sum_{cyc} a(a+c).$$

Now we compute $\sum_{cyc} a(a+c)=(a^2+b^2+c^2)+(ab+bc+ca)$. Hence

$$\sum_{cyc} a\sqrt[n]{a^n+c^n}\ge 2^{\frac{1}{n}-1}(a^2+b^2+c^2+ab+bc+ca).$$

We want to prove $\sum_{cyc} a\sqrt[n]{a^n+c^n}\ge\sqrt[n]{2}(ab+bc+ca)$, so it suffices to show

$$2^{\frac{1}{n}-1}(a^2+b^2+c^2+ab+bc+ca)\ge 2^{\frac{1}{n}}(ab+bc+ca).$$

Actually, dividing both sides by $2^{\frac{1}{n}-1}>0$, this is equivalent to

$$a^2+b^2+c^2+ab+bc+ca\ge 2(ab+bc+ca)\Leftrightarrow a^2+b^2+c^2\ge ab+bc+ca,$$

which trivially holds. Equality occurs when $a=b=c$, and the proof is complete.

Solution 2 by proposer

Lemma.

If $a,b,c>0$ and $n\in\mathbb{N}, n\ge 2$ then:

$$a\sqrt[n]{b^n+c^n}\ge\frac{a(b+c)}{2}\cdot\sqrt[n]{2}.$$

Proof.

$b^n+c^n\overset{Holder}{\ge}\frac{(b+c)^n}{2^{n-1}}\Rightarrow a\sqrt[n]{b^n+c^n}\ge\frac{a(b+c)}{2}\cdot\sqrt[n]{2}$, with equality for $b=c$.

Let's get back to the main problem. Using the Lemma we obtain:

$$LHS=\sum a\sqrt[n]{b^n+c^n}\overset{Lemma}{\ge}\sum\frac{a(b+c)}{2}\cdot\sqrt[n]{2}=\sqrt[n]{2}(ab+bc+ca)=RHS.$$

Equality holds if and only if $a=b=c$.

Note: For $n=2$ we obtain the proposed problem in RMM 4/2024

If $a,b,c>0$ then:

$$\sum a\sqrt{b^2+c^2}\ge\sqrt{2}(ab+bc+ca)$$

Daniel Sitaru

SP.593 Solve for real numbers

$$(\sin x+\cos y)^2=(\sin x+1)(\cos y-1)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Jose Luis Diaz – Barrero – Spain

Let $a=\sin x, b=\cos y$.

Then the equation becomes

$$(a+b)^2=(a+1)(b-1).$$

Expanding both sides yields $a^2+2ab+b^2=ab-a+b-1$, and rearranging terms gives

$$a^2+ab+b^2+a-b+1=0\Leftrightarrow a^2+(b+1)a+(b^2-b+1)=0.$$

This quadratic in a has discriminant $\Delta=(b+1)^2-4(b^2-b+1)=-3(b-1)^2\le 0$, and the equation has real solutions only when $\Delta=0$, which occurs precisely when $b=1$.

For $b=1$, we obtain

$$a^2+2a+1=(a+1)^2=0\Rightarrow a=-1,$$

and therefore, $\sin x=-1$, $\cos y=1$. From which we obtain the full set of solutions:

$$x=-\frac{\pi}{2}+2k\pi,\qquad y=2m\pi,\qquad k,m\in\mathbb{Z}.$$

Solution 2 by proposer

$$(\sin x+\cos y)^2=(\sin x+1)(\cos y-1)$$
$$\sin^2 x+2\sin x\cos y+\cos^2 y=\sin x\cos y-\sin x+\cos y-1=0$$
$$\sin^2 x+\sin x\cos y+\sin x+\cos^2 y-\cos y+1=0$$

Denote: $\sin x=a;\ a\in\mathbb{R}$

$$a^2+a(\cos y+1)+\cos^2 y-\cos y+1=0$$
$$\Delta=(\cos y+1)^2-4(\cos^2 y-\cos y+1)=$$
$$=\cos^2 y+2\cos y+1-4\cos^2 y+4\cos y-4=$$
$$=-3\cos^2 y+6\cos y-3=$$
$$=-3(\cos^2 y-2\cos y+1)=-3(\cos y-1)^2$$
$$a\in\mathbb{R}\Rightarrow\Delta\geq 0\Rightarrow -3(\cos y-1)^2\geq 0\Rightarrow$$
$$\Rightarrow\cos y-1=0\Rightarrow\cos y=1$$
$$y\in\{2k\pi|k\in\mathbb{Z}\}$$
$$a=\frac{-(\cos y+1)}{2}=\frac{-(1+1)}{2}=\frac{-2}{2}=-1$$
$$\sin x=-1$$
$$x\in\left\{(-1)^k\left(-\frac{\pi}{2}\right)+k\pi|k\in\mathbb{Z}\right\},\qquad x\in\left\{(-1)^{k+1}\cdot\frac{\pi}{2}+k\pi|k\in\mathbb{Z}\right\}$$

SP.594 Find $x,y>0$ such that

$$\ln^2(xy)=\ln(xe)\cdot\ln\left(\frac{y}{e}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Jose Luis Diaz – Barrero – Spain

Let $a=\ln x$, $b=\ln y$. Then:

$$\ln(xy)=a+b,\ln(xe)=a+1,\ln\left(\frac{y}{e}\right)=b-1.$$

Substituting into the equation gives $(a+b)^2=(a+1)(b-1)$. Expanding both sides and arranging terms, we obtain the quadratic equation in a:

$$a^2+(b+1)a+(b^2-b+1)=0.$$

Its discriminant is $\Delta=(b+1)^2-4(b^2-b+1)=-3(b-1)^2$. Since $\Delta\geq 0$ is required for real a, we must have $b=1$. Then $\Delta=0$, and

$$a=-\frac{b+1}{2}=-1.$$

Thus $\ln x=-1\Rightarrow x=e^{-1}$ and $\ln y=1\Rightarrow y=e$. Therefore, the unique solution with $x,y>0$ is

$$(x,y)=(e^{-1},e).$$

Solution 2 by proposer

$$\ln^2(xy)=\ln(xe)\cdot\left(\ln\frac{y}{e}\right)$$

$$(\ln x+\ln y)^2=(\ln x+\ln e)(\ln y-\ln e)$$

$$(\ln x+\ln y)^2=(\ln x+1)(\ln y-1)$$

$$\ln^2 x+\ln^2 y+2\ln x\ln y=\ln x\ln y-\ln x+\ln y-1$$

$$\ln^2 x+\ln x\ln y+\ln x+\ln^2 y-\ln y+1=0$$

Denote: $a=\ln x\,;a\in\mathbb{R}$

$$\ln^2 x+\ln x\,(1+\ln y)+(\ln^2 y-\ln y+1)=0$$

$$a^2+a(1+\ln y)+(\ln^2 y-\ln y+1)=0$$

$$a\in\mathbb{R}\Rightarrow\Delta\geq 0\Rightarrow(1+\ln y)^2-4(\ln^2 y-\ln y+1)\geq 0$$

$$1+2\ln y+\ln^2 y-4\ln^2 y+4\ln y-4\geq 0$$

$$-3\ln^2 y+6\ln y-3\geq 0\Rightarrow -3(\ln^2 y-2\ln y+1)\geq 0$$

$$\Rightarrow -3(\ln^2 y-1)^2\geq 0\Rightarrow \ln y-1=0$$

$$\ln y=1\Rightarrow y=e$$

$$a=\frac{-(1+\ln y)}{2}=\frac{-1(1+1)}{2}=-\frac{2}{2}=-1$$

$$\ln x=-1\Rightarrow \ln x=\ln\frac{1}{e}\Rightarrow x=\frac{1}{e}$$

SP.595 Solve for real numbers:

$$\tan 2x + \tan 3x + \tan 5x = \tan 2x \cdot \tan 3x \cdot \tan 5x$$

Proposed by Daniel Sitaru – Romania

Solution by Jose Luis Diaz – Barrero – Spain

Using the well-known identity $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, we substitute it into the formula

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.$$

When $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ the numerator is zero, so $\tan(A+B+C) = 0$, which implies

$$A + B + C = k\pi, \qquad k \in \mathbb{Z}$$

In our problem, we set $A = 2x, B = 3x, C = 5x$, and we have $A + B + C = 10x$.

By the preceding identity, we have

$$10x = k\pi, k \in \mathbb{Z} \Leftrightarrow x = \frac{k\pi}{10}, k \in \mathbb{Z}.$$

Next, we must exclude those values of x for which any of $\tan 2x$, $\tan 3x$, or $\tan 5x$ is undefined. Recall that $\tan t$ is undefined when

$$t = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}.$$

- Condition for $\tan 2x$ to be undefined. Substituting $x = \frac{k\pi}{10}$ in $2x = \frac{\pi}{2} + n\pi$ gives $k = \frac{5(2n+1)}{2}$. This is never an integer, so there is no integer k for which $\tan 2x$ is undefined. Thus $\tan 2x$ imposes no restriction on k.

- Condition for $\tan 3x$ to be undefined. Substituting $x = \frac{k\pi}{10}$ in $3x = \frac{\pi}{2} + n\pi$ gives $k = \frac{5+10n}{3}$. Thus, $\tan 3x$ is undefined exactly when $k \equiv 5 \pmod{10}$. So, these values of k must be excluded.

- Condition for $\tan 5x$ to be undefined. Substituting $x = \frac{k\pi}{10}$ in $5x = \frac{\pi}{2} + n\pi$ gives

$k = 2n + 1$. So $\tan 5x$ is undefined exactly when k is odd. Therefore, all odd integers k must be excluded.

Notice that every integer of the form $5 + 10t$ is odd, so the restriction from $\tan 3x$ is already contained in the restriction k odd. Hence the only condition we need is that k be even. Thus, we take only even integers k. Let $k = 2m, m \in \mathbb{Z}$. Then

$$x = \frac{k\pi}{10} = \frac{2m\pi}{10} = \frac{m\pi}{5}, \qquad m \in \mathbb{Z}.$$

For such x, all of $\tan 2x$, $\tan 3x$, and $\tan 5x$ are defined, and the original equation holds.

Therefore, the complete solution set is

$$x = \frac{m\pi}{5}, \qquad m \in \mathbb{Z}.$$

SP.596 **Solve for real numbers:**

$$\begin{cases} \sin^2 x = \dfrac{1}{2} + \sin^2(y - z) \\ \sin^2 y = \dfrac{1}{3} + \sin^2(z - x) \\ \sin^2 z = \dfrac{1}{6} + \sin^2(x - y) \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

The system can be written:

$$\begin{cases} \dfrac{1 - \cos 2x}{2} = \dfrac{1}{2} + \dfrac{1 - \cos 2(y - z)}{2} \\ \dfrac{1 - \cos 2y}{2} = \dfrac{1}{3} + \dfrac{1 - \cos 2(z - x)}{2} \\ \dfrac{1 - \cos 2z}{2} = \dfrac{1}{6} + \dfrac{1 - \cos 2(x - y)}{2} \end{cases}, \qquad \begin{cases} -\cos 2x = 1 - \cos 2(y - z) \\ -\cos 2y = \dfrac{2}{3} - \cos 2(z - x) \\ -\cos 2z = \dfrac{1}{3} - \cos 2(x - y) \end{cases}$$

$$\begin{cases} \cos 2(y - z) - \cos 2x = 1 \\ \cos 2(z - x) - \cos 2y = \dfrac{2}{3} \\ \cos 2(x - y) - \cos 2z = \dfrac{1}{3} \end{cases}, \qquad \begin{cases} 2\sin\dfrac{2(y - z) + 2x}{2}\sin\dfrac{2x - 2(y - z)}{2} = 1 \\ 2\sin\dfrac{2(z - x) + 2y}{2}\sin\dfrac{2y - 2(z - x)}{2} = \dfrac{2}{3} \\ 2\sin\dfrac{2(z - x) + 2z}{2}\sin\dfrac{2z - 2(x - y)}{2} = \dfrac{1}{3} \end{cases}$$

$$\begin{cases}\sin(x+y-z)\sin(x-y+z)=\frac{1}{2}\\ \sin(-x+y+z)\sin(x+y-z)=\frac{1}{3}\\ \sin(x-y+z)\sin(-x+y+z)=\frac{1}{6}\end{cases}$$

By multiplying:

$$(\sin(x+y-z)\sin(x-y+z)\sin(-x+y+z))^2=\frac{1}{36}$$

$$\sin(x+y-z)\sin(x-y+z)\sin(-x+y+z)=\pm\frac{1}{6}$$

$$\begin{cases}\frac{1}{2}\sin(-x+y+z)=\pm\frac{1}{6}\\ \frac{1}{3}\sin(x-y+z)=\pm\frac{1}{6},\\ \frac{1}{6}\sin(x+y-z)=\pm\frac{1}{6}\end{cases}\qquad \begin{cases}\sin(-x+y+z)=\pm\frac{1}{3}\\ \sin(x-y+z)=\pm\frac{1}{2}\\ \sin(x+y-z)=\pm 1\end{cases}$$

$$\begin{cases}-x+y+z=(-1)^m\arcsin\left(\pm\frac{1}{3}\right)+m\pi & (1)\\ x-y+z=(-1)^n\arcsin\left(\pm\frac{1}{2}\right)+n\pi & (2)\\ x+y-z=(-1)^p\arcsin(\pm 1)+\pi & (3)\end{cases},m,n,p\in\mathbb{Z}$$

By adding (1); (2):

$$2z=(-1)^m\arcsin\left(\pm\frac{1}{3}\right)+(-1)^n\arcsin\left(\pm\frac{1}{2}\right)+(m+n)\pi$$

$$z=\frac{1}{2}\left((-1)^m\arcsin\left(\pm\frac{1}{3}\right)+(-1)^n\arcsin\left(\pm\frac{1}{2}\right)+(m+n)\pi\right)$$

By adding (1); (3):

$$2y=(-1)^m\arcsin\left(\pm\frac{1}{3}\right)+(-1)^p\arcsin(\pm 1)+(m+p)\pi$$

$$y=\frac{1}{2}\left((-1)^m\arcsin\left(\pm\frac{1}{3}\right)+(-1)^p\arcsin(\pm 1)+(m+p)\pi\right)$$

By adding (2); (3):

$$2x=(-1)^n\arcsin\left(\pm\frac{1}{2}\right)+(-1)^p\arcsin(\pm 1)+(n+p)\pi$$

$$x=\frac{1}{2}\left((-1)^n\arcsin\left(\pm\frac{1}{2}\right)+(-1)^p\arcsin(\pm 1)+(n+p)\pi\right)$$

SP.597 Let a,b,c,d be positive real numbers with $\sum a \ge \sum \frac{1}{a}$. Prove that:

$$\sum \frac{a+b+c-d}{a^4+b^4+c^4+abcd} \le \frac{4}{3}\left(\frac{ab+ac+ad+bc+bd+cd}{abc+abd+acd+bcd}\right)$$

Proposed by Huseyin Yigit Emekci – Izmir – Turkey

Solution by proposer

Note that by majorization $(4,0,0) \succ (2,1,1)$. Hence, by Muirhead's inequality

$$a^4+b^4+c^4 \ge abc(a+b+c)$$

Using this we obtain

$$\sum \frac{a+b+c-d}{a^4+b^4+c^4+abcd} \le \sum \frac{a+b+c-d}{abc(a+b+c)+abcd}$$

$$= \frac{1}{abcd\sum a}\left[\sum d(a+b+c-d)\right]$$

$$= \frac{2(ab+ac+ad+bc+bd+cd)-\sum a^2}{abcd\sum a}$$

On the other hand, note that from Power Mean and Maclaurin's inequalities

$$\sum a^2 \overset{P-M}{\ge} \frac{(\sum a)^2}{4} \text{ and } \frac{\sum a}{4} \ge \sqrt{\frac{ab+ac+ad+bc+bd+cd}{6}}$$

Implies that $\sum a^2 \ge 2(ab+ac+ad+bc+bd)$ / 3. Then

$$\sum \frac{a+b+c-d}{a^4+b^4+c^4+abcd} \le \frac{2(ab+ac+ad+bc+bd+cd)-\sum a^2}{abcd\sum a} \le$$

$$\le \frac{4(ab+ac+ad+bc+bd+cd)}{3abcd\sum a}$$

Finally, we need to show

$$\frac{4(ab+ac+ad+bc+bd+cd)}{3abcd\sum a} \ge \frac{4}{3}\left(\frac{ab+ac+ad+bc+bd+cd}{abc+abd+acd+bcd}\right)$$

which is equivalent to $\sum a \ge \sum \frac{1}{a}$, which was our problem condition. Equality holds for

$a=b=c=d=1$.

SP.598 Let x, y, z be positive real numbers. Prove that:

$$\frac{x^3+9xy^2}{z^3+x^2y}+\frac{y^3+9yz^2}{x^3+y^2z}+\frac{z^3+9zx^2}{y^3+z^2x}\geq 3+\frac{12xyz(x+y+z)}{x^3y+y^3z+z^3x}$$

Proposed by Huseyin Yigit Emekci-Turkiye

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{x^3+9xy^2}{z^3+x^2y}+\frac{y^3+9yz^2}{x^3+y^2z}+\frac{z^3+9zx^2}{y^3+z^2x}=$$

$$\frac{x^4}{z^3x+x^3y}+\frac{y^4}{x^3y+y^3z}+\frac{z^4}{y^3z+z^3x}+\frac{9x^2y^2}{z^3x+x^3y}+\frac{9y^2z^2}{x^3y+y^3z}+\frac{9z^2x^2}{y^3z+z^3x}$$

$$\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}x^2\right)^2}{2\sum_{cyc}x^3y}+\frac{9\left(\sum_{cyc}xy\right)^2}{2\sum_{cyc}x^3y}$$

$$=\frac{\left(\sum_{cyc}x^2\right)^2+9\left(\sum_{cyc}xy\right)^2}{2\sum_{cyc}x^3y}\overset{?}{\geq}\frac{3\sum_{cyc}x^3y+12xyz\left(\sum_{cyc}x\right)}{\sum_{cyc}x^3y}$$

$$\Leftrightarrow\sum_{cyc}x^4+11\sum_{cyc}x^2y^2-6xyz\left(\sum_{cyc}x\right)-6\sum_{cyc}x^3y\underset{(*)}{\overset{?}{\geq}}0$$

Now, if $P(1,1,1)=0$ and $1+A+B+C+D=0$, then:

$$P(x,y,z)=\sum_{cyc}x^4+A\sum_{cyc}x^2y^2+Bxyz\left(\sum_{cyc}x\right)+C\sum_{cyc}x^3y+D\sum_{cyc}xy^3\geq 0$$

holds iff: $3(1+A)\geq C^2+CD+D^2\to ①$ and in our case,

with reference to the LHS of $(*)$, say $F(x,y,z)$, we have: $F(1,1,1)=0$ and

$A=11, B=-6, C=-6$ and $D=0$; and so, $3(1+A)-(C^2+CD+D^2)$

$=3(1+11)-(-6)^2=0\ \therefore$ via ①, we see that $(*)$ is true

$$\therefore\frac{x^3+9xy^2}{z^3+x^2y}+\frac{y^3+9yz^2}{x^3+y^2z}+\frac{z^3+9zx^2}{y^3+z^2x}\geq 3+\frac{12xyz(x+y+z)}{x^3y+y^3z+z^3x}\ \forall x,y,z>0,$$

'' ='' iff $x=y=z$ (QED)

Solution 2 by proposer

Note that left hand side of the inequality is

$$LHS=\sum_{cyc}\frac{x^3}{z^3+x^2y}+9\sum_{cyc}\frac{xy^2}{z^3+x^2y}=\sum_{cyc}\frac{x^4}{z^3x+x^3y}+9\sum_{cyc}\frac{x^2y^2}{z^3x+x^3y}$$

Using Cauchy – Schwarz, we obtain

$$LHS=\sum_{cyc}\frac{x^4}{z^3x+x^3y}+9\sum_{cyc}\frac{x^2y^2}{z^3x+x^3y}\geq\frac{(x^2+y^2+z^2)^2+9(xy+yz+zx)^2}{2(x^3y+y^3z+z^3x)}$$

$$=\frac{x^4+y^4+z^4+11(x^2y^2+y^2z^2+z^2x^2)+18xyz(x+y+z)}{2(x^3y+y^3z+z^3x)}$$

On the other hand, the right hand side of the inequality is

$$RHS=\frac{3(x^3y+y^3z+z^3x)+12xyz(x+y+z)}{x^3y+y^3z+z^3x}$$

Hence we need to show that

$$\frac{x^4+y^4+z^4+11(x^2y^2+y^2z^2+z^2x^2)+18xyz(x+y+z)}{2(x^3y+y^3z+z^3x)}\geq$$

$$\geq\frac{3(x^3y+y^3z+z^3x)+12xyz(x+y+z)}{x^3y+y^3z+z^3x}$$

or

$$x^4+y^4+z^4+11(x^2y^2+y^2z^2+z^2x^2)+18xyz(x+y+z)\geq$$

$$\geq 6(x^3y+y^3z+z^3x)+24xyz(x+y+z)$$

or

$$x^4+y^4+z^4+11(x^2y^2+y^2z^2+z^2x^2)\geq 6[x^3y+y^3z+z^3x+xyz(x+y+z)]\quad \textbf{(1)}$$

On the other hand, this inequality is equivalent to

$$\frac{1}{2}\sum\nolimits_{cyc}(x^2-y^2-2xy+4yz-2xz)^2\geq 0\qquad \textbf{(2)}$$

which is true. The equality holds for $x=y=z$.

Remark. The equivalence of inequalities (1) and (2) have been obtained in the book "Algebraic Inequalities: Old and New Methods". One may refer to it.

SP.599 **We consider the function $f: D \to \mathbb{R}$**

$$f(x) = x\int_x^{x+\frac{3}{x}} t \arcsin\left(\frac{1}{t}\right) dt$$

where D is the maximal domain of the function.

a. Find the domain D

b. Show that the function $f(x)$ is even

c. Calculate $\lim_{x \to -\infty} f(x)$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

a. The domain D

We consider the function:

$$g(t) = t \arcsin\left(\frac{1}{t}\right)$$

In order to determine the maximal domain of this function, we impose the following conditions:

$$\frac{1}{t} \geq -1 \text{ and } \frac{1}{t} \leq 1$$

We obtain:

$$t \in (-\infty, -1] \cup [1, \infty)$$

We also notice that the function $g(t)$ is even, because $g(-t) = g(t)$.

We have to decide now the range of variation of the variable x (so the domain of the function $f(x)$.)

For $t \in [1, \infty)$ we must have $x \in [1, \infty)$.

For $t \in (-\infty, -1]$ we must have $x \in (-\infty, -1]$

So the maximal domain of the function $f(x)$ is:

$$D = (-\infty, -1] \cup [1, \infty)$$

b. We prove that the function $f(x)$ is even

Because the function $g(t)$ is even, for any $x \in [1, \infty)$ we can write:

$$\int_x^{x+\frac{3}{x}} g(t)\,dt = \int_{-\left(x+\frac{3}{x}\right)}^{-x} g(t)\,dt = -\int_{-x}^{-\left(x+\frac{3}{x}\right)} g(t)\,dt$$

So:

$$f(-x) = (-x)\int_{-x}^{-\left(x+\frac{3}{x}\right)} g(t)\,dt = (-x)\left(-\int_x^{x+\frac{3}{x}} g(t)\,dt\right) = f(x)$$

So the function $f(x)$ is even.

c. We calculate the limit required in the statement of the problem

We calculate $\lim_{x\to-\infty} f(x)$

Because the function $f(x)$ is even, we have:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x)$$

We can write using the mean value theorem:

$$\int_x^{x+\frac{3}{x}} t \arcsin\left(\frac{1}{t}\right) dt = \left(x + \frac{3}{x} - x\right) \frac{\arcsin\left(\frac{1}{c}\right)}{\frac{1}{c}} = \frac{3}{x} \cdot \frac{\arcsin\left(\frac{1}{c}\right)}{\frac{1}{c}}$$

where $x < c < x + \frac{3}{x}$

When $x \to \infty$, we have $c \to \infty$.

So:

$$\lim_{x\to\infty} f(x) = \lim_{c\to\infty} 3 \cdot \frac{\arcsin\left(\frac{1}{c}\right)}{\frac{1}{c}} = 3$$

SP.600 Let a, b, c, d, e, f, g be real numbers such that

$a \ge b \ge c \ge d \ge e \ge f \ge g$ and $a + b + c + d + e + f + g = 0.$

Prove that:

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 \ge 2(ab + bc + cd + de + ef + fg + ga)$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Write the inequality in the homogeneous form $E(a,b,c,d,e,f,g)\ge 0$, where

$$E(a,b,c,d,e,f,g)=a^2+b^2+c^2+d^2+e^2+f^2+g^2-$$

$$-2(ab+bc+cd+de+ef+fg+ga)+\frac{(a+b+c+d+e+f+g)^2}{7}$$

Let

$$x=\frac{a+b}{2},\ \ y=\frac{f+g}{2}$$

Since

$$E(a,b,c,d,e,f,g)-E(x,x,c,d,e,f,g)=(a^2+b^2-2x^2)+2(x^2-ab)+2(x-b)c+$$
$$+2(x-a)g$$

$$=\frac{(a-b)^2}{2}+\frac{(a-b)^2}{2}+(a-b)c-(a-b)g=(a-b)^2+(a-b)(c-g)\ge 0$$

and

$$E(a,b,c,d,e,f,g)-E(a,b,c,d,e,y,y)=(f-g)^2+(f-g)(a-e)\ge 0,$$

we have

$$E(a,b,c,d,e,f,g)\ge E(x,x,c,d,e,y,y).$$

So, it suffices to show that

$$E(x,x,c,d,e,y,y)\ge 0.$$

We have

$$7E(x,x,c,d,e,y,y)=7(2x^2+c^2+d^2+e^2+2y^2)-$$
$$-14(x^2+cx+cd+de+ey+y^2+xy)+$$
$$+(2x+c+d+e+2y)^2=4x^2-2(5c-2d-2e+3y)x+A,$$

where

$$A=7(c^2+d^2+e^2+2y^2)+(c+d+e+2y)^2-14(cd+de+ey+y^2)$$
$$=8(c^2+d^2+e^2)-12cd+2ce-12de+4y^2+4cy+4dy-10ey.$$

Therefore,

$$7E(x,x,c,d,e,y,y)=\left(2x-\frac{5c-2d-2e+3y}{2}\right)^2+A-\left(\frac{5c-2d-2e+3y}{2}\right)^2$$

$$=\left(2x-\frac{5c-2d-2e+3y}{2}\right)^2+\frac{7(y-c+2d-2e)^2}{4}\geq 0.$$

The equality occurs for $a=b:=x, f=g:=y, 2x+c+d+e+2y=0$,

$4x-5c+2d+2e-3y=0$ and $y-c+2d-2e=0$, i.e. for

$(a,b,c,d,e,f,g)=(x,x,-y,-x-y,-x,y,y)$. From

$x\geq x\geq -y\geq -x-y\geq -x\geq y\geq y$, we get $y=-x$, hence the equality occurs for

$(a,b,c,d,e,f,g)=(x,x,x,0,-x,-x,-x)$ with $x\geq 0$.

UNDERGRADUATE PROBLEMS

UP.586 If $A\in M_{2,1}(\mathbb{R}); B\in M_{1,2}(\mathbb{R}); A\cdot B=\begin{pmatrix}0 & 0\\ 8 & 1\end{pmatrix}$ then find $B\cdot A$.

Proposed by Daniel Sitaru -Romania

Solution by proposer

$$(A\cdot B)^2=(A\cdot B)(A\cdot B)=\begin{pmatrix}0 & 0\\ 8 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 8 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 8 & 1\end{pmatrix}=A\cdot B$$

$$(A\cdot B)^2=A\cdot B \qquad (1)$$

By (1):

$$(A\cdot B)^2=A\cdot B\Rightarrow B\cdot(A\cdot B)^2=B\cdot(A\cdot B)\Rightarrow$$

$$\Rightarrow B\cdot(A\cdot B)^2\cdot A=B\cdot(A\cdot B)\cdot A$$

$$B\cdot(A\cdot B)\cdot(A\cdot B)\cdot A=B\cdot(A\cdot B)\cdot A$$

$$(B\cdot A)\cdot(BA)\cdot(BA)=(B\cdot A)\cdot(B\cdot A)$$

$$(B\cdot A)^3=(B\cdot A)^2 \qquad (2)$$

$$rank\ (B\cdot A)\geq rank\ (A\cdot(B\cdot A)\cdot B)=$$

$$=rank\ (AB)^2\overset{(1)}{=}rank\ (A\cdot B)=rank\begin{pmatrix}0 & 0\\ 8 & 1\end{pmatrix}=1$$

$$A\in M_{2,1}(\mathbb{R}), B\in M_{1,2}(\mathbb{R})\Rightarrow B\cdot A\in M_{1,1}(\mathbb{R})$$

$$\left.\begin{array}{r}rank\ (B\cdot A)\geq 1\\ B\cdot A\in M_{1,1}(\mathbb{R})\end{array}\right\}\Rightarrow (B\cdot A)=1\Rightarrow \det(B\cdot A)\neq 0\Rightarrow (\exists)(B\cdot A)^{-1}$$

By (2):

$$(B \cdot A)^3 = (B \cdot A)^2 \Rightarrow (B \cdot A)^3 \cdot (B \cdot A)^{-1} = (B \cdot A)^2 \cdot (B \cdot A)^{-1}$$

$$(B \cdot A)^2 = (B \cdot A) \Rightarrow (B \cdot A)^2 \cdot (B \cdot A)^{-1} = (B \cdot A) \cdot (B \cdot A)^{-1} \Rightarrow B \cdot A = (1)$$

UP.587 **Let a, b, c be positive real numbers such that at most one of them is less than $ab + bc + ca = 3$. Prove that:**

$$abc(a + b + c)^3 \leq 27.$$

Proposed by Vasile Cîrtoaje – Romania

Solution 1 by Luis Diaz – Barrero – Spain

$$F(a, b, c) = abc(a + b + c)^3$$

under the constraint $ab + bc + ca = 3, a, b, c > 0$, and assume that at most one of a, b, c is less than 1. Since at most one of a, b, c is less than 1, at least two of them are ≥ 1. Without loss of generality, assume $a \geq 1, b \geq 1$. From the constraint, $3 = ab + bc + ca \geq ab$ we get $ab \leq 3$. Because $a, b \geq 1$, it follows that

$$1 \leq a \leq 3, \qquad 1 \leq b \leq 3.$$

Next, write the constraint as $ab + c(a + b) = 3$. Using $ab \geq 1$ (since $a, b \geq 1$), we obtain $3 = ab + c(a + b) \geq 1 + c(a + b)$, hence $c(a + b) \leq 2$. Since $a, b \geq 1$, we have $a + b \geq 2$ and therefore $c \leq \frac{2}{a+b} \leq 1$. Thus $0 < c \leq 1$.

Consequently, every point (a, b, c) in the domain satisfies

$$1 \leq a \leq 3, \qquad 1 \leq b \leq 3, \qquad 0 < c \leq 1.$$

Hence the domain is contained in the bounded set $[1, 3] \times [1, 3] \times (0, 1]$. Since F is continuous, it is bounded above on this set, so F is bounded above on the given domain. Consider the closed box $K = [1, 3] \times [1, 3] \times [0, 1]$ and the closed set $S = \{(a, b, c) \in K |\ ab + bc + ca = 3\}$. Then S is closed in K, hence compact. Our actual domain is $D = \{(a, b, c) \in S | c > 0\}$, which is a relatively open subset of S. Since F is continuous on S, it attains its maximum on S at some point $(a_0, b_0, c_0) \in S$. If $c_0 = 0$, then from $ab + bc + ca = 3$ we get $a_0 b_0 = 3$ and

$$F(a_0, b_0, c_0) = a_0 b_0 c_0 (a_0 + b_0 + c_0)^3 = 0.$$

But on D we have $c > 0$, so $F > 0$ there; hence the maximum of F on S cannot occur at a point with $c = 0$. Therefore the maximum on S is attained at some point $(a_0, b_0, c_0) \in D$, with $c_0 > 0$. Thus F attains its maximum on the domain D.

We now find the critical points of F under the constraint $ab + bc + ca = 3$ with $a, b, c > 0$. To do that we consider the Lagrangian

$$\mathcal{L}(a, b, c, \lambda) = abc(a + b + c)^3 - \lambda(ab + bc + ca - 3).$$

Taking partial derivatives and setting them to zero, we obtain

$$\frac{\partial \mathcal{L}}{\partial a} = bc(a + b + c)^3 + 3abc(a + b + c)^2 - \lambda(b + c) = 0,$$
$$\frac{\partial \mathcal{L}}{\partial b} = ac(a + b + c)^3 + 3abc(a + b + c)^2 - \lambda(a + c) = 0,$$
$$\frac{\partial \mathcal{L}}{\partial c} = ab(a + b + c)^3 + 3abc(a + b + c)^2 - \lambda(a + b) = 0,$$

together with the constraint $ab + bc + ca = 3$. Subtracting the first two equations gives

$$bc(a + b + c)^3 - ac(a + b + c)^3 - \lambda(b + c - a - c) = 0 \Leftrightarrow$$
$$\Leftrightarrow c(b - a)(a + b + c)^3 - \lambda(b - a) = 0.$$

If $b \neq a$, we can divide by $(b - a)$ and obtain

$$c(a + b + c)^3 = \lambda.$$

Likewise, if $b \neq c$ and $c \neq a$ we obtain $a(a + b + c)^3 = \lambda$, and $b(a + b + c)^3 = \lambda$, respectively. Thus, if a, b, c are not all equal, we would get $a = b = c$, a contradiction.

Therefore any interior critical point must satisfy $a = b = c$. Imposing the constraint $ab + bc + ca = 3$ with $a = b = c$ gives $a = 1$ (since $a > 0$).

Hence the only critical point in D is

$$(a, b, c) = (1, 1, 1).$$

At this point, $F(1, 1, 1) = 27$. Finally, we can now conclude the following:

- The function F is bounded above on the domain D and attains its maximum there.
- Any interior maximizer must be a critical point of the Lagrangian, hence must satisfy $a = b = c = 1$.

Therefore the maximum of F on the domain

$$\{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\in\mathbb{R}^3:\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}>0, ab+bc+ca=3, at\ most\ on\ of\ \boldsymbol{a},\boldsymbol{b},\boldsymbol{c}<1\}$$

is $\max F = 27$, attained uniquely at $a = b = c = 1$. Equality holds when $a = b = c = 1$, and this completes the proof.

Solution 2 by proposer

Assume that $a \geq b \geq 1 \geq c$. Denoting $x = \sqrt[3]{a}, y = \sqrt[3]{b}$ and $z = \sqrt[3]{c}$, we need to show that

$$\frac{3}{xyz} \geq x^3 + y^3 + z^3$$

for $x \geq y \geq 1 \geq z > 0$ such that $x^3y^3 + y^3z^3 + z^3x^3 = 3$. From

$$3 = x^3y^3 = y^3z^3 + z^3x^3 \geq 3y^3z^3,$$

we get $yz \leq 1$. For fixed x, assume that z is a function of y and write the desired inequality as $F(y) \geq 0$, where

$$F(y) = \frac{3}{xyz} - x^3 - y^3 - z^3.$$

By differentiating the equality constraint, we get

$$z^2(x^3 + y^3)z' + y^2(x^3 + z^3) = 0,\ \ z' = \frac{-y^2(x^3 + z^3)}{z^2(x^3 + y^3)}.$$

Therefore,

$$\frac{F'(y)}{3} = \left(\frac{-1}{xyz^2} - z^2\right)z' - \frac{1}{xy^2z} - y^2 = \frac{y(xyz^4 + 1)(x^3 + z^3)}{xz^4(x^3 + y^3)} - \frac{xy^4z + 1}{xy^2z}.$$

We will show that $F'(y) \geq 0$, that is

$$\frac{y^3(xyz^4 + 1)}{z^3(xy^4z + 1)} \geq \frac{x^3 + y^3}{x^3 + z^3},\qquad \frac{y^3(xyz^4 + 1)}{z^3(xy^4z + 1)} - 1 \geq \frac{x^3 + y^3}{x^3 + z^3} - 1,$$

$$\frac{y^3 - z^3}{z^3(xy^4z + 1)} \geq \frac{y^3 - z^3}{x^3 + z^3},\quad \frac{1}{z^3(xy^4z + 1)} \geq \frac{1}{x^3 + z^3}.$$

The last inequality reduce to $x^2 \geq y^4z^4$, which is true because $x \geq 1$ and $yz \leq 1$. Since $F'(y) \geq 0, F(y)$ is increasing and has the minimum value when y is minimum, hence when $y = 1$. So, it suffices to consider the case $y = 1$, when we need to show that $x^3 + z^3 + x^3z^3 = 3$ implies

$$3 \geq xz(x^3 + z^3 + 1).$$

Indeed, we have

$$3 - xz(x^3 + z^3 + 1) = 3 - xz(4 - x^3z^3) = x^4z^4 - 4xz + 3 =$$
$$= (xz - 1)^2(x^2z^2 + 2xz + 3) \geq 0.$$

The equality occurs when $a = b = c = 1$.

UP.588 If $a \geq 0$ then:

$$15\left(\int_0^a \frac{x}{e^x}dx \cdot \int_0^a \frac{x^2}{e^x}dx \cdot \int_0^a \frac{x^3}{e^x}dx\right)^2 \leq a^9\left(\int_0^a \frac{x^2}{e^{2x}}dx\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Via the Integral form of CBS inequality:}$$

$$15\left(\int_0^a \frac{x}{e^x}dx.\int_0^a \frac{x^2}{e^x}dx.\int_0^a \frac{x^3}{e^x}dx\right)^2$$

$$= 15\left(\int_0^a 1.\frac{x}{e^x}dx\right)^2.\left(\int_0^a x.\frac{x}{e^x}dx\right)^2.\left(\int_0^a x^2.\frac{x}{e^x}dx\right)^2$$

$$\leq 15\left(\int_0^a 1\,dx.\int_0^a \frac{x^2}{e^{2x}}dx\right).\left(\int_0^a x^2\,dx.\int_0^a \frac{x^2}{e^{2x}}dx\right).\left(\int_0^a x^4\,dx.\int_0^a \frac{x^2}{e^{2x}}dx\right)$$

$$= 15.[x]_0^a.\left[\frac{x^3}{3}\right]_0^a.\left[\frac{x^5}{5}\right]_0^a.\left(\int_0^a \frac{x^2}{e^{2x}}dx\right)^3 = 15(a)\left(\frac{a^3}{3}\right)\left(\frac{a^5}{5}\right).\left(\int_0^a \frac{x^2}{e^{2x}}dx\right)^3$$

$$= a^9\left(\int_0^a \frac{x^2}{e^{2x}}dx\right)^3 \therefore 15\left(\int_0^a \frac{x}{e^x}dx.\int_0^a \frac{x^2}{e^x}dx.\int_0^a \frac{x^3}{e^x}dx\right)^2 \leq a^9\left(\int_0^a \frac{x^2}{e^{2x}}dx\right)^3 \ \forall\, a \geq 0,$$

$$'' ='' \ \text{iff } a = 0 \text{ (QED)}$$

Solution 2 by proposer

By CBS – integral form:

$$\left(\int_0^a \frac{x}{e^x}\right)^2 = \left(\int_0^a 1 \cdot \frac{x}{e^x}dx\right)^2 \overset{CBS}{\leq} \left(\int_0^a 1^2\,dx\right)\left(\int_0^a \left(\frac{x}{e^x}\right)^2 dx\right) = a \cdot \int_0^a \frac{x^2}{e^{2x}}dx$$

$$\left(\int_0^a \frac{x}{e^x}\right)^2 \le a\cdot\int_0^a \frac{x^2}{e^{2x}}dx \qquad (1)$$

$$\left(\int_0^a \frac{x^2}{e^x}dx\right)^2 = \left(\int_0^a x\cdot\frac{x}{e^x}dx\right)^2 \le \left(\int_0^a x^2dx\right)\cdot\left(\int_0^a \left(\frac{x}{e^x}\right)^2dx\right) = \frac{a^3}{3}\cdot\int_0^a \frac{x^2}{e^{2x}}dx$$

$$\left(\int_0^a \frac{x^2}{e^x}dx\right)^2 \le \frac{a^3}{3}\int_0^a \frac{x^2}{e^{2x}}dx$$

$$3\left(\int_0^a \frac{x^2}{e^x}dx\right)^2 \le a^3\int_0^a \frac{x^2}{e^{2x}}dx \qquad (2)$$

$$\left(\int_0^a \frac{x^3}{e^x}dx\right)^2 = \left(\int_0^a x^2\cdot\frac{x}{e^x}dx\right)^2 \le \left(\int_0^a x^4dx\right)\left(\int_0^a \left(\frac{x}{e^x}\right)^2dx\right) = \frac{a^5}{5}\left(\int_0^a \frac{x^2}{e^{2x}}dx\right)$$

$$5\left(\int_0^a \frac{x^3}{e^x}dx\right)^2 \le a^5\left(\int_0^a \frac{x^2}{e^{2x}}dx\right) \qquad (3)$$

By multiplying (1); (2); (3):

$$15\left(\int_0^a \frac{x}{e^x}dx\cdot\int_0^a \frac{x^2}{e^x}dx\cdot\int_0^a \frac{x^3}{e^x}dx\right)^2 \le a^9\left(\int_0^a \frac{x^2}{e^{2x}}dx\right)^3$$

Equality holds for $a = 0$.

UP.589 **If $X, Y, Z \in M_4(\mathbb{C})$ are matrices such that:**

$$\begin{cases} X = 2Y + Z \\ X^2 = 4Y + 4Z \\ X^3 = 8Y + 12Z \end{cases} \quad \text{then: } X^{2024} = 2^{2024}\cdot Y + 2024\cdot 2^{2023}\cdot Z$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\begin{cases} X = 2Y + Z \\ X^2 = 4Y + 4Z \end{cases} \Rightarrow \begin{cases} -4X = -8Y - 4Z \\ X^2 = 4Y + 4Z \end{cases}$$

$$X^2 - 4X = 4Y + 4Z - 8Y - 4Z$$

$$X^2 - 4X = -4Y \qquad (1)$$

$$\begin{cases} X^2 = 4Y + 4Z \\ X^3 = 8Y + 12Z \end{cases} \Rightarrow \begin{cases} -3X^2 = -12Y - 12Z \\ X^3 = 8Y + 12Z \end{cases}$$

$$X^3 - 3X^2 = 8Y + 12Z - 12Y - 12Z$$

$$X^3 - 3X^2 = -4Y \qquad (2)$$

By (1); (2): $X^3 - 3X^2 = X^2 - 4X,\ \ X^3 - 4X^2 + 4X = O_4$ (3)

By multiplying (3) with x^{n-2}:

$$X^{n+1} - 4X^n + 4X^{n-1} = O_4 \quad (4)$$

We will prove by mathematical induction:

$$P(n): X^n = 2^nY + n \cdot 2^{n-1}Z$$

For $n = 1: X = 2Y + Z$ (true by hypothesis)

$$P(n): X^n = 2^nY + n \cdot 2^{n-1} \cdot Z \quad \text{(suppose true)}$$

$$P(n+1): X^{n+1} = 2^{n+1}Y + (n+1) \cdot 2^n \cdot Z \quad \text{(to prove)}$$

By (4):

$$X^{n+1} = 4X^n - 4X^{n-1} \overset{P(n)}{=}$$

$$= 4(2^nY + n \cdot 2^{n-1} \cdot z) - 4(2^{n-1}Y + (n-1) \cdot 2^{n-2} \cdot z) =$$

$$= 2^{n+2}Y + n \cdot 2^{n+1}Z - 2^{n+1}Y + (n-1)2^n \cdot Z =$$

$$= 2^{n+1}Y(2-1) + 2^n \cdot Z(2n - n + 1) = 2^{n+1}Y + n \cdot 2^n \cdot Z$$

$$P(n) \to P(n+1)$$

For $n = 2024$ in $P(n)$: $X^{2024} = 2^{2024}Y + 2024 \cdot 2^{2023} \cdot Z$

UP.590 If $A, B \in M_4(\mathbb{R})$; $A \cdot B = B \cdot A$ then:

$$\det\left(A^4 + B^4 + AB(A^2 + AB + B^2)\right) \geq 0$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Let's consider the equation $z^5 = 1$ for complex numbers. The roots are:

$$z_k = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}; k \in \overline{0,4}$$

$$z^5 - 1 = 0 \Rightarrow (z-1)(z^4 + z^3 + z^2 + z + 1) = 0$$

$z_0 = 1$ hence z_1, z_2, z_3, z_4 are solutions for the equation:

$$z^4 + z^3 + z^2 + z + 1 = 0$$

$$z^4 + z^3 + z^2 + z + 1 = (z - z_1)(z - z_2)(z - z_3)(z - z_4) \quad (1)$$

$$z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}; z_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$$

$$z_3=\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5};z_4=\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}$$

$$z_3=\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}=\cos\left(\pi+\frac{\pi}{5}\right)+i\sin\left(\pi+\frac{\pi}{5}\right)=$$

$$=-\cos\frac{\pi}{5}-i\sin\frac{\pi}{5}=-\cos\left(\pi-\frac{4\pi}{5}\right)-i\sin\left(\pi-\frac{4\pi}{5}\right)=$$

$$=\cos\frac{4\pi}{5}-i\sin\frac{4\pi}{5}=\overline{z_2}$$

$$z_4=\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}=\cos\left(2\pi-\frac{2\pi}{5}\right)+i\sin\left(2\pi-\frac{2\pi}{5}\right)=$$

$$=\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}=\overline{z_1}$$

$$z_3=\overline{z_2};z_4=\overline{z_1}$$

By (1):

$$z^4+z^3+z^2+z+1=(z-z_1)(z-\overline{z_1})(z-z_2)(z-\overline{z_2})$$

$$(A-z_1B)(A-\overline{z_1}B)=A^2-\overline{z_1}AB-z_1BA+z_1\overline{z_1}B^2=$$

$$=A^2-(z_1+\overline{z_1})AB+z_1\overline{z_1}B^2 \quad (2)$$

$$z_1\overline{z_1}=\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)\left(\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}\right)=$$

$$=\cos^2\frac{2\pi}{5}+\sin^2\frac{2\pi}{5}=1$$

$$z_1+\overline{z_1}=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}=2\cos\frac{2\pi}{5}=$$

$$=2\left(2\cos^2\frac{\pi}{5}-1\right)=2\cdot\left(2\cdot\left(\frac{\sqrt{5}+1}{4}\right)^2-1\right)=$$

$$=2\cdot\left(2\cdot\frac{5+1+2\sqrt{5}}{16}-1\right)=2\cdot\left(\frac{6+2\sqrt{5}}{8}-1\right)=$$

$$=2\cdot\frac{2\sqrt{5}-2}{8}=\frac{\sqrt{5}-1}{2}$$

By (2):

$$(A-z_1B)(A-\overline{z_1}B)=A^2=\frac{\sqrt{5}-1}{2}AB+B^2 \quad (3)$$

$$(A-z_2B)(A-\overline{z_2}B)=A^2-z_2BA-\overline{z_2}AB+z_2\overline{z_2}B^2=$$

$$= A^2 - (z_2 + \overline{z_2})AB + z_2\overline{z_2}B^2 \quad (4)$$

$$z_2\overline{z_2} = \left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)\left(\cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}\right) =$$

$$= \cos^2\frac{4\pi}{5} + \sin^2\frac{4\pi}{5} = 1$$

$$z_2 + \overline{z_2} = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} + \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5} =$$

$$= 2\cos\frac{4\pi}{5} = 2\cos\left(\pi - \frac{\pi}{5}\right) = -2\cos\frac{\pi}{5} =$$

$$= -2\cdot\frac{\sqrt{5}+1}{4} = -\frac{\sqrt{5}+1}{2}$$

By (4):

$$(A - z_2B)(A - \overline{z_2}B) = A^2 + \frac{\sqrt{5}+1}{2}AB + B^2 \quad (5)$$

By (3); (5):

$$(A - z_1B)(A - \overline{z_1}B)(A - z_2B)(A - \overline{z_2}B) =$$

$$= \left(A^2 + B^2 - \frac{\sqrt{5}-1}{2}AB\right)\left(A^2 + B^2 + \frac{\sqrt{5}+1}{2}AB\right) =$$

$$= (A^2 + B^2)^2 + AB(A^2 + B^2)\cdot\left(\frac{\sqrt{5}+1}{2} - \frac{\sqrt{5}-1}{2}\right) - \frac{(\sqrt{5}-1)(\sqrt{5}+1)}{4}A^2B^2 =$$

$$= (A^2 + B^2) + AB(A^2 + B^2)\cdot\frac{2}{2} - \frac{4}{4}A^2B^2 = A^4 + B^4 + 2A^2B^2 + A^3B + AB^3 - A^2B^2 =$$

$$= A^4 + A^3B + A^2B^2 + AB^3 + B^4 = A^4 + B^4 + AB(A^2 + AB + B^2)$$

$$A^4 + B^4 + AB(A^2 + AB + B^2) =$$

$$= (A - z_1B)(A - \overline{z_1}B)(A - z_2B)(A - \overline{z_2}B)$$

$$\det\left(A^4 + B^4 + AB(A^2 + AB + B^2)\right) =$$

$$= \det\left((A - z_1B)(A - \overline{z_1}B)(A - z_2B)(A - \overline{z_2}B)\right) =$$

$$= \det(A - z_1B)\cdot\det(A - \overline{z_1}B)\cdot\det(A - z_2B)\ \det(A - \overline{z_2}B) =$$

$$= \det(A - z_1B)\cdot\overline{\det(A - z_1B)}\cdot\det(A - z_2B)\cdot\overline{\det(A - z_2B)} =$$

$$= |\det(A - z_1B)|^2\cdot|\det(A - z_2B)|^2 \ge 0$$

UP.591 Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{3^n} \sum_{k=0}^{n-1} \sqrt{\binom{n}{k}\binom{n}{k+1}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Jose Luis Diaz – Barrero – Spain

For $0 \le k \le n-1$ we use the identity

$$\binom{n}{k+1} = \binom{n}{k}\frac{n-k}{k+1}.$$

Therefore,

$$\sqrt{\binom{n}{k}\binom{n}{k+1}} = \binom{n}{k}\sqrt{\frac{n-k}{k+1}}.$$

Thus the sum becomes

$$S_n = \sum_{k=0}^{n-1} \sqrt{\binom{n}{k}\binom{n}{k+1}} = \sum_{k=0}^{n-1} \binom{n}{k}\sqrt{\frac{n-k}{k+1}}.$$

We now observe that for all k,

$$\frac{n-k}{k+1} \le n, \text{ and therefore } \sqrt{\frac{n-k}{k+1}} \le \sqrt{n}.$$

Hence,

$$S_n \le \sqrt{n}\sum_{k=0}^{n-1}\binom{n}{k} \le \sqrt{n}\sum_{k=0}^{n}\binom{n}{k} = \sqrt{n}2^n.$$

Consequently,

$$0 \le \frac{S_n}{3^n} \le \frac{\sqrt{n}2^n}{3^n} = \sqrt{n}\left(\frac{2}{3}\right)^n.$$

Since

$$\sqrt{n}\left(\frac{2}{3}\right)^n \to 0 \quad \text{as} \quad n \to \infty,$$

the squeeze theorem implies that the limit exists and its value is

$$\Omega = \lim_{n\to\infty}\frac{1}{3^n}\sum_{k=0}^{n-1}\sqrt{\binom{n}{k}\binom{n}{k-1}} = 0.$$

Solution 2 by proposer

$$\sum_{k=0}^{n-1}\sqrt{\binom{n}{k}\binom{n}{k+1}} \overset{AM-GM}{\leq} \sum_{k=0}^{n-1}\frac{1}{2}\left(\binom{n}{k}+\binom{n}{k+1}\right) =$$

$$= \frac{1}{2}\left(\sum_{k=0}^{n-1}\binom{n}{k}+\sum_{k=0}^{n-1}\binom{n}{k+1}\right) =$$

$$= \frac{1}{2}\left(\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}-1\right)+\frac{1}{2}\left(\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}-1\right) =$$

$$= \frac{1}{2}(2^n-1)+\frac{1}{2}(2^n-1) = 2^n-1$$

$$0 < \sum_{k=0}^{n-1}\sqrt{\binom{n}{k}\binom{n}{k+1}} \leq 2^n-1$$

$$0 < \frac{1}{3^n}\sum_{k=0}^{n-1}\sqrt{\binom{n}{k}\binom{n}{k+1}} \leq \frac{2^n-1}{3^n}$$

$$0 \leq \lim_{n\to\infty}\frac{1}{3^n}\sum_{k=0}^{n-1}\sqrt{\binom{n}{k}\binom{n}{k+1}} \leq \lim_{n\to\infty}\frac{2^n-1}{3^n}$$

$$0 < \Omega \leq 0$$

$$\Omega = 0$$

UP.592 Solve for real numbers:

$$\begin{cases}\cos x+\cos y+\cos z = 1\\ \cos^2 x+\cos^2 y+\cos^2 z = 1\\ \cos^3 x+\cos^3 y+\cos^3 z = 1\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\sum_{cyc}\cos x = 1 \Rightarrow \left(\sum_{cyc}\cos x\right)^2 = 1 \Rightarrow$$

$$\sum_{cyc}\cos^2 x+2\sum_{cyc}\cos x\cos y=1\Rightarrow 1=2\sum_{cyc}\cos x\cos y=1$$

$$\sum_{cyc}\cos x\cos y=0\qquad (1)$$

$$1\cdot 1-1=0\Rightarrow\left(\sum_{cyc}\cos x\right)\left(\sum_{cyc}\cos^2 x\right)-\sum_{cyc}\cos^3 x=0$$

$$\sum_{cyc}\cos^3 x+\sum_{cyc}\cos x\,(\cos^2 y+\cos^2 z)-\sum_{cyc}\cos^3 x=0$$

$$\sum_{cyc}\cos x\,(\cos^y+\cos^2 z)=0\qquad (2)$$

By (1); (2) and $1\cdot 0-0=0$

$$\sum_{cyc}\cos x\cdot\sum_{cyc}\cos x\cos y-\sum_{cyc}\cos x\,(\cos^2 y+\cos^2 z)=0$$

$$\sum_{cyc}\cos^2 x\cos y+\sum_{cyc}\cos x\cos^2 y+3\cos x\cos y\cos z-$$

$$-\sum_{cyc}\cos x\cos^2 y-\sum_{cyc}\cos x\cos^2 z=0,\qquad 3\cos x\cos y\cos z=0$$

$$\begin{cases}\cos x+\cos y+\cos z=1\\ \cos x\cos y\cos z=0\end{cases}\Rightarrow\begin{cases}\cos x=0\\ \cos y=0\\ \cos z=1\end{cases}\text{or}\begin{cases}\cos x=0\\ \cos y=1\\ \cos z=0\end{cases}\text{or}\begin{cases}\cos x=1\\ \cos y=0\\ \cos z=0\end{cases}$$

Solution: $x=\pm\frac{\pi}{2}+2k\pi;k\in\mathbb{Z};y=\pm\frac{\pi}{2}+2m\pi;m\in\mathbb{Z},\ z=2n\pi;n\in\mathbb{Z}$

and permutations.

UP.593 For $b\geq a$, prove that:

$$\int_a^b\frac{(x+1)^3-3x}{e^{x^3}}dx\leq b-a+\ln\left(\frac{b^3+1}{a^3+1}\right)$$

with equality if and only if $a=b$.

Proposed by Huseyin Yigit Emekci – Izmir – Turkey

Solution by proposer

Observe that for $a=b$, we have the equality case $LHS=RHS=0$. On the other hand,

by definition $e^x=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ and Bernoulli Inequality (*)

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \overset{Bernoulli}{\geq} 1+x$$

we can derive the well – known inequality $e^x \geq x+1$ for $x \geq -1$ easily as above.

However, for $x < -1$ define $f(x) = e^x - x - 1$.

$$f'(x) = e^x - 1 = 0 \Rightarrow \text{critical point is } x = 0$$

Implies that $x = 0$ is a global minimum. Thus, the inequality is true. Hence, we have

$e^{x^3} \geq x^3 + 1$. Then

$$\int_a^b \frac{(x+1)^3 - 3x}{e^{x^3}}dx \leq \int_a^b \frac{(x+1)^3 - 3x}{x^3+1}$$

Then, proceeding and solving the integral gives

$$\int_a^b \frac{(x+1)^3 - 3x}{e^{x^3}}dx \leq \int_a^b \frac{(x+1)^3 - 3x}{x^3+1}dx = \int_a^b 1\,dx + \int_a^b \frac{3x^2}{x^3+1}dx$$

$$= b - a + \ln(b^3+1) - \ln(a^3+1) = b - a + \ln\left(\frac{b^3+1}{a^3+1}\right)$$

as desired. Equality holds just for $a = b$.

References. We have referred to a not well-known inequality theorem in (*). Here is the Bernoulli's inequality:

For any real number $x \geq -1, r \geq 1$, we have

$$(1+x)^r \geq 1 + rx$$

with equality if and only if $r = 1$.

UP.594 Solve the system

$$\begin{cases} x - 2y + z + 2 = k^2, \quad with\ 3 < k < 11 \\ x^2 + y^2 + z^2 = 109659 \\ -x^4 + y^2 + z^2 = 80929 \\ 3 < x < y < z, \quad x, y, z \in \mathbb{N}. \end{cases}$$

Proposed by Said Attaoui – Oran – Algeria

Solution by proposer

We are given the system

$$\begin{cases} x-2y+z+2=k^2, \quad with\ 3<k<11 \\ x^2+y^2+z^2=109659 \\ -x^4+y^2+z^2=80929 \\ 3<x<y<z, \quad x,y,z\in\mathbb{N}. \end{cases}$$

Subtracting the third equation from the second eliminates y^2+z^2

$$x^2+x^4=109659-80929=28730$$

Trying integer values $x>3$, we find

$$x=13\Rightarrow 13^2+13^4=169+28561=28730$$

Thus, $x=13$ is the unique solution for x. Substituting into the second equation gives

$$y^2+z^2=109659-169=109490$$

From the first equation

$$13-2y+z+2=k^2\Rightarrow z=2y-15+k^2$$

Substituting into the equation $y^2+z^2=109490$ gives

$$y^2+(2y-15+k^2)^2=109490\Rightarrow 5y^2+4(k^2-15)y+(k^2-15)^2=109490$$

This is quadratic in y with integer coefficients, valid for $k\in\{4,5,6,7,8,9,10\}$. For example,

- For $k=9$, this yields $y=121, z=308$
- For $k=10$, we get $y=113, z=311$

In both cases, we verify

$$x^2+y^2+z^2=109659, \qquad -x^4+y^2+z^2=80929, \quad x-2y+z+2=k^2$$

and that $3<x<y<z$. Hence, they are valid solutions.

Finally,

$$(x,y,z)=(13,113,311) \quad \text{and} \quad (x,y,z)=(13,121,308)$$

As conclusion, the system uniquely determines $x=13$, as it is the only positive integer satisfying the equation $x^4+x^2=28730$. Once x is fixed, the system reduces to a Diophantine equation in y and z, constrained by:

$$y^2+z^2=109490 \ \text{ and } \ z=2y-15+k^2, \quad \text{for } k\in\{4,5,\dots,10\}.$$

For each admissible integer k, this leads to a quadratic in y with integer coefficients. In several cases – such as $k = 9$ and $k = 10$ – this discriminant is a perfect square, yielding integer solutions for y, and hence z.

This proves not only that solutions exist, but also that *multiple distinct integer triples of the form* $(13, y, z)$ satisfy the entire system under the given constraints. Thus, the system admits *multiple valid solutions*, all with $x = 13$ and with different pairs (y, z) determined by suitable values of k between 4 and 10.

UP.595 We consider the function $u: \mathbb{R} \to \mathbb{R}$, periodic with period 2π. For the period $[0, 2\pi]$ we have: $u(x) = 0$ if $x \in \left[0, \frac{\pi}{2}\right)$; $u(x) = -\cos(x)$ if $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$; $u(x) = 0$ if $x \in \left[\frac{3\pi}{2}, 2\pi\right)$. Prove the equality:

$$\int_0^\infty \frac{u(x)}{1 + x^2} dx = -\frac{\pi}{4e} + \frac{e^2 + 1}{2e} \arctan\left(\frac{1}{e}\right)$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote:

$$I = \int_0^\infty \frac{u(x)}{1 + x^2} dx$$

The function $u(x)$ satisfies Dirichlet's conditions. Also, the function is even.

We expand the function in the Fourier series:

$$u(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)$$

where:

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} u(x)\, dx; A_n = \frac{1}{\pi} \int_0^{2\pi} u(x) \cos(nx)\, dx$$

Calculating these integrals, we obtain:

$$A_0=\frac{1}{\pi};A_1=-\frac{1}{2};A_n=\frac{-1}{(n^2-1)\pi}\cdot\left[\cos\left(\frac{n\pi}{2}\right)+\cos\left(\frac{3n\pi}{2}\right)\right]\text{ for } n\geq 2$$

But:

$\cos\left(\frac{3n\pi}{2}\right)=\cos\left(\frac{n\pi}{2}\right)$ for any $n\geq 2$

Result:

$$A_n=\frac{-1}{\pi}\cdot\left[\frac{2\cos\left(\frac{n\pi}{2}\right)}{n^2-1}\right]=\frac{-1}{\pi}\cdot\left[\frac{\cos\left(\frac{n\pi}{2}\right)}{n-1}-\frac{\cos\left(\frac{n\pi}{2}\right)}{n+1}\right]=B_n+C_n$$

Where

$$B_n=\frac{-1}{\pi}\cdot\frac{\cos\left(\frac{n\pi}{2}\right)}{n-1};\ \ C_n=\frac{1}{\pi}\cdot\frac{\cos\left(\frac{n\pi}{2}\right)}{n+1}$$

We have:

$B_n=0$ if $n=3,5,7,\dots;B_n\neq 0$ if $n=2,4,6,\dots;$

$C_n=0$ if $n=3,5,7,\dots;\ C_n\neq 0$ if $n=2,4,6,\dots;$

So:

$$u(x)=\frac{1}{\pi}-\frac{1}{2}\cos(x)+[B_2\cos(2x)+B_4\cos(4x)+B_6\cos(6x)+B_8\cos(8x)+\cdots]+$$
$$+[C_2\cos(2x)+C_4\cos(4x)+C_6\cos(6x)+C_8\cos(8x)+\cdots]$$

$$u(x)=\frac{1}{\pi}-\frac{1}{2}\cos(x)+\left[\frac{1}{\pi}\cos(2x)-\frac{1}{3\pi}\cos(4x)+\frac{1}{5\pi}\cos(6x)-\frac{1}{7\pi}\cos(8x)+\cdots\right]+$$
$$+\left[-\frac{1}{3\pi}\cos(2x)+\frac{1}{5\pi}\cos(4x)-\frac{1}{7\pi}\cos(6x)+\frac{1}{9\pi}\cos(8x)+\cdots\right]$$

We now use the following relationship:

$$\int_0^\infty\frac{\cos(mx)}{1+x^2}dx=\frac{\pi}{2}e^{-m},\text{ where } m>0$$

This relation is Laplace's integral and is well known.

It is easily proved for example using the properties of the Laplace transform.

We obtained the value of the integral I:

$$I=\frac{1}{\pi}\frac{\pi}{2}-\frac{1}{2}\frac{\pi}{2}e^{-1}+\left(\frac{1}{\pi}\frac{\pi}{2}e^{-2}-\frac{1}{3\pi}\frac{\pi}{2}e^{-4}+\frac{1}{5\pi}\frac{\pi}{2}e^{-6}-\frac{1}{7\pi}\frac{\pi}{2}e^{-8}+\cdots\right)+$$
$$+\left(-\frac{1}{3\pi}\frac{\pi}{2}e^{-2}+\frac{1}{5\pi}\frac{\pi}{2}e^{-4}-\frac{1}{7\pi}\frac{\pi}{2}e^{-6}+\frac{1}{9\pi}\frac{\pi}{2}e^{-8}+\cdots\right)$$

$$2I = 1 - \frac{\pi}{2}e^{-1} + \left(e^{-2} - \frac{1}{3}e^{-4} + \frac{1}{5}e^{-6} - \frac{1}{7}e^{-8} + \cdots\right) +$$

$$+\left(-\frac{1}{3}e^{-2} + \frac{1}{5}e^{-4} - \frac{1}{7}e^{-6} + \frac{1}{9}e^{-8} + \cdots\right)$$

We will now use the power series development of the following functions

$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$ **where** $-1 \le x \le 1$

$x\arctan(x) = x^2 - \frac{1}{3}x^4 + \frac{1}{5}x^6 - \frac{1}{7}x^8 + \cdots$ **where** $-1 \le x \le 1$

$\frac{\arctan(x)}{x} = 1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{1}{7}x^6 + \cdots$ **where** $-1 \le x \le 1$

We have:

$$e^{-1}\arctan(e^{-1}) = e^{-2} - \frac{1}{3}e^{-4} + \frac{1}{5}e^{-6} - \frac{1}{7}e^{-8} + \cdots$$

$$\frac{\arctan(e^{-1})}{e^{-1}} = 1 - \frac{1}{3}e^{-2} + \frac{1}{5}e^{-4} - \frac{1}{7}e^{-6} + \cdots$$

We obtained:

$$2I = -\frac{\pi}{2}e^{-1} + e^{-1}\arctan(e^{-1}) + \frac{\arctan(e^{-1})}{e^{-1}}$$

Or:

$$I = -\frac{\pi}{4e} + \frac{e^2+1}{2e}\arctan\left(\frac{1}{e}\right)$$

UP.596 If $x > 0, y > 0, z > 0$ prove that there exists $u > 0$ such as

$$\frac{\sin x \sin y + \sin y \sin z + \sin z \sin x}{xy + yz + zx} = \frac{\sin u}{u}$$

Proposed by Cristian Miiu – Romania

Solution by proposer

It is easy to prove that:

$$\min\left(\frac{\sin x \sin y}{xy}, \frac{\sin y \sin z}{yz}, \frac{\sin z \sin x}{zx}\right) \le \frac{\sum \sin x \sin y}{\sum xy} \le$$

$$\le \max\left(\frac{\sin x \sin y}{xy}, \frac{\sin y \sin z}{yz}, \frac{\sin z \sin x}{zx}\right)$$

Now using Cauchy theorem for $f: [x-y, x+y] \to \mathbb{R}$

$f(t)=\cos t$ and $g:[x-y,x+y]\to\mathbb{R};g(t)=t^2$ we obtain that there exist

$c_1\in(x-y,x+y)$ such as

$$\frac{\cos(x+y)-\cos(x-y)}{(x+y)^2-(x-y)^2}=\frac{-\sin c_1}{2c_2}\leftrightarrow\frac{\sin x\sin y}{xy}=\frac{\sin c_1}{c_1}$$

In the same way

$$\frac{\sin y\sin z}{yz}=\frac{\sin c_2}{c_2}\text{ and }\frac{\sin z\sin x}{zx}=\frac{\sin c_3}{c_3}$$

Now

$$\min\left(\frac{\sin c_1}{c_1},\frac{\sin c_2}{c_2},\frac{\sin c_3}{c_3}\right)\le\frac{\sum\sin x\sin y}{\sum xy}\le\max\left(\frac{\sin c_1}{c_1},\frac{\sin c_2}{c_2},\frac{\sin c_3}{c_3}\right)$$

But $\alpha\to\frac{\sin t}{\alpha}$ is continuous so that exists $u>0$ such as

$$\sin u=\frac{\sum\sin x\sin y}{\sum xy}$$

UP.597 Find the following limit:

$$L=\lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\lim_{x\to\frac{\pi}{n}}\left(\sum_{k=0}^{n}\binom{n}{k}\sin(k+1)x\right)\right)$$

Proposed by Marian Ursărescu and Florică Anastase – Romania

Solution 1 by Jose Luis Diaz – Barrero – Spain

We will use a complex representation. Indeed, consider

$$\sum_{k=0}^{n}\binom{n}{k}e^{i(k+1)x}=e^{ix}\sum_{k=0}^{n}\binom{n}{k}e^{ikx}=e^{ix}(1+e^{ix})^n.$$

Taking imaginary parts, we get

$$\sum_{k=0}^{n}\binom{n}{k}\sin((k+1)x)=\mathcal{T}(e^{ix}(1+e^{ix})^n),$$

and $L=\lim_{n\to\infty}\frac{1}{2^n}\mathcal{T}\left(e^{i\frac{\pi}{n}}\left(1+e^{i\frac{\pi}{n}}\right)^n\right)$. Since $e+e^{i\frac{\pi}{n}}=e^{i\frac{\pi}{2n}}\left(e^{-i\frac{\pi}{2n}}+e^{i\frac{\pi}{2n}}\right)=$

$2\cos\left(\frac{\pi}{2n}\right)e^{i\frac{\pi}{2n}}$, then $\left(1+e^{i\frac{\pi}{n}}\right)^n=2^n\cos^n\left(\frac{\pi}{2n}\right)e^{i\frac{\pi}{2}}=2^n\cos^n\left(\frac{\pi}{2n}\right)i$, and therefore

$$\frac{1}{2^n}\mathcal{T}\left(e^{i\frac{\pi}{n}}\left(1+e^{i\frac{\pi}{n}}\right)^n\right)=\cos^n\left(\frac{\pi}{2n}\right)\mathcal{T}\left(ie^{i\frac{\pi}{n}}\right).$$

Since $ie^{i\frac{\pi}{n}} = i\left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right) = -\sin\frac{\pi}{n} + i\cos\frac{\pi}{n}$, its imaginary part is $\cos\left(\frac{\pi}{n}\right)$.

Thus

$$L = \lim_{n\to\infty} \cos^n\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{n}\right).$$

Since, $\cos\left(\frac{\pi}{n}\right) \to 1$ when $n \to \infty$, and $\cos\left(\frac{\pi}{2n}\right) = 1 - \frac{\pi^2}{8n^2} + o(n^{-2})$, then

$$\ln\left(\cos^n\left(\frac{\pi}{2n}\right)\right) = n\ln\left(1 - \frac{\pi^2}{8n^2} + o(n^{-2})\right) \to 0,$$

and

$$\cos^n\left(\frac{\pi}{2n}\right) \to 1.$$

Therefore $L = 1$, and we are done.

Solution 2 by proposers

Let be $S_1 = \sum_{k=0}^{n}\binom{n}{k}\cos(k+1)\,x$ and $S_2 = \sum_{k=0}^{n}\binom{n}{k}\sin(k+1)x$

Using the identity

$$\sum_{k=1}^{n}\binom{n}{k} z^{k+1} = z(1+z)^n$$

for $z = \cos x + i\sin x$, we get:

$$S_1 + iS_2 = (\cos x + i\sin x)(\cos x + i\sin x + 1)^n \qquad (1)$$

But $\cos x + i\sin x + 1 = 2\cos^2\frac{x}{2} + 2i\sin\frac{x}{2}\cos\frac{x}{2} = 2\cos\frac{x}{2}\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)$, then (1) becomes:

$$S_1 + iS_2 = 2^n\cos^n\frac{x}{2}\left(\cos\frac{n+2}{2}x + i\sin\frac{n+2}{2}x\right) \qquad (2)$$

By develop and identifying in (2), to obtain:

$$\begin{cases} S_1 = \sum_{k=0}^{n}\binom{n}{k}\cos(k+1)x = 2^n\cos^n\frac{x}{2}\cos\frac{n+2}{2}x \\ S_2 = \sum_{k=0}^{n}\binom{n}{k}\sin(k+1)\,x = 2^n\cos^n\frac{x}{2}\sin\frac{n+2}{2}x \end{cases}$$

Therefore,

$$L=\lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\lim_{x\to\frac{\pi}{n}}\left(\sum_{k=0}^{n}\binom{n}{k}\sin(k+1)x\right)\right)=$$

$$=\lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\lim_{n\to\infty}\left(2^n\cos^n\frac{x}{2}\sin\frac{n+2}{2}x\right)\right)=\lim_{n\to\infty}\cos^n\frac{\pi}{2n}\sin\frac{n+2}{n}\cdot\frac{\pi}{2}=1$$

UP.598 Find the following limit:

$$L=\lim_{x\to 0}\left(\frac{1}{x}\cdot\lim_{x\to\infty}\sum_{k=1}^{n}3^{k-1}\sin^3\frac{x}{3^k}\right),x\in\mathbb{R}.$$

Proposed by Marian Ursărescu and Florică Anastase – Romania

Solution 1 by Jose Luis Diaz – Barrero – Spain

To compute the limit we use the well-known identity

$$\sin^3 t=\frac{3\sin t-\sin(3t)}{4}.$$

Putting $t=\frac{x}{3^k}$ we have $\sin^3\left(\frac{x}{3^k}\right)=\frac{3\sin\left(\frac{x}{3^k}\right)-\sin\left(\frac{x}{3^{k-1}}\right)}{4}$ or

$$3^{k-1}\sin^3\left(\frac{x}{3^k}\right)=\frac{3^k}{4}\sin\left(\frac{x}{3^k}\right)-\frac{3^{k-1}}{4}\sin\left(\frac{x}{3^{k-1}}\right),$$

after multiplication by 3^{k-1}. Let us denote $a_k=\frac{3^{k-1}}{4}\sin\left(\frac{x}{3^{k-1}}\right)$, then the preceding expression equals $a_{k+1}-a_k$, and the sum in the statement telescopes.

That is,

$$\sum_{k=1}^{n}3^{k-1}\sin^3\left(\frac{x}{3^k}\right)=a_{n+1}-a_1=\frac{3^n}{4}\sin\left(\frac{x}{3^n}\right)-\frac{1}{4}\sin x.$$

Since $\lim_{n\to\infty}\left(\frac{x}{3^n}\right)=0$, then $\lim_{n\to\infty}3^n\sin\left(\frac{x}{3^n}\right)=\lim_{n\to\infty}\frac{x}{\left(\frac{x}{3^n}\right)}\sin\left(\frac{x}{3^n}\right)=x$, and we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{n}3^{k-1}\sin^3\left(\frac{x}{3^k}\right)=\frac{x-\sin x}{4}.$$

Thus, the outer limit is

$$L=\lim_{x\to 0}\frac{1}{x}\cdot\frac{x-\sin x}{4}=\frac{1}{4}\lim_{x\to 0}\left(1-\frac{\sin x}{x}\right)=0,$$

on accout that $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$, as it is well-known.

Solution 2 by proposers

Using the identity $\sin 3a = 3\sin a - 4\sin^3 a$ for $a \in \left\{\frac{x}{3}, \frac{x}{3^2}, \dots, \frac{x}{3^{n-1}}\right\}$ we get:

$$\begin{cases} \sin x = 3\sin\frac{x}{3} - 4\sin^3\frac{x}{3} \\ \sin\frac{x}{3} = 3\sin\frac{x}{3^2} - 4\sin^3\frac{x}{3^2} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \\ \sin\frac{x}{3^{n-1}} = 3\sin\frac{x}{3^n} - 4\sin^3\frac{x}{3^n} \end{cases}$$

By multiplying with $1, 3, 3^2, \dots, 3^{n-1}$ and summing, it follows:

$$\sum_{k=1}^{n} 3^{k-1}\sin^3\frac{x}{3^k} = \frac{1}{4}\left(3^n\sin\frac{x}{3^n} - \sin x\right)$$

and then

$$\lim_{n\to\infty}\sum_{k=1}^{n} 3^{k-1}\sin^3\frac{x}{3^k} = \frac{1}{4}(x - \sin x)$$

Therefore,

$$L = \lim_{x\to 0}\left(\frac{1}{x}\cdot\lim_{n\to\infty}\sum_{k=1}^{n} 3^{k-1}\sin^3\frac{x}{3^k}\right) =$$

$$= \lim_{x\to 0}\frac{1}{4x}(x - \sin x) = \frac{1}{4}\lim_{x\to 0}\left(1 - \frac{\sin x}{x}\right) = 0$$

UP.599 Calculate the integral:

$$\int_{-\pi}^{\pi}\frac{\operatorname{arccot}(x)}{\sqrt{3-\cos(x)}}dx$$

In this problem we will consider that definition of the function $\operatorname{arccot}(x)$ which has the image the interval $(0,\pi)$.

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote:

$$A=\int_{-\pi}^{\pi}\frac{\operatorname{arccot}(x)}{\sqrt{3-\cos(x)}}dx;\ B=\int_{-\pi}^{\pi}\frac{\arctan(x)}{\sqrt{3-\cos(x)}}dx$$

We have:

$$A+B=\frac{\pi}{2}\int_{-\pi}^{\pi}\frac{1}{\sqrt{3-\cos(x)}}dx=\frac{\pi}{2}\cdot 2\cdot\int_{0}^{\pi}\frac{1}{\sqrt{3-\cos(x)}}dx=\pi\int_{0}^{\pi}\frac{1}{\sqrt{3-\cos(x)}}dx$$

because the function to be integrated is even.

But:

$$B=\int_{-\pi}^{\pi}\frac{\arctan(x)}{\sqrt{3-\cos(x)}}dx=0$$

because the function to be integrate is odd.

We obtain:

$$A=\pi\int_{0}^{\pi}\frac{1}{\sqrt{3-\cos(x)}}dx$$

We consider the integral I:

$$I=\int_{0}^{\pi}\frac{1}{\sqrt{3-\cos(x)}}dx$$

We have:

$$3-\cos(x)=2+2\sin^2\left(\frac{x}{2}\right),\qquad I=\frac{1}{\sqrt{2}}\int_{0}^{\pi}\frac{1}{\sqrt{1+\sin^2\left(\frac{x}{2}\right)}}dx$$

In the integral I we make the variable change $u=\sin\left(\frac{x}{2}\right)$

We obtain:

$$I=\sqrt{2}\int_{0}^{1}\frac{1}{\sqrt{1-u^4}}du$$

We make the variable change $t=u^4$

We obtain:

$$I=\frac{\sqrt{2}}{4}\int_{0}^{1}t^{-\frac{3}{4}}(1-t)^{-\frac{1}{2}}dt$$

We consider the integral J:

$$J=\int_0^1 t^{-\frac{3}{4}}(1-t)^{-\frac{1}{2}}dt$$

We will use the Euler's Beta function:

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt$$

We set the conditions:

$$p-1=-\frac{3}{4};\quad q-1=-\frac{1}{2}$$

Result:

$$p=\frac{1}{4};\quad q=\frac{1}{2}$$

We obtain:

$$J=B\left(\frac{1}{4},\frac{1}{2}\right)$$

$$J=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}=\frac{\Gamma^2\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}=\frac{\Gamma^2\left(\frac{1}{4}\right)\sqrt{\pi}}{\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}}=\frac{1}{\sqrt{2}\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

where $\Gamma(\alpha)$ is the Euler's Gamma function.

Result:

$$I=\frac{1}{4\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

We obtained the value of the integral A:

$$A=\pi\cdot\frac{1}{4\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

$$A=\frac{1}{4}\sqrt{\pi}\Gamma^2\left(\frac{1}{4}\right)$$

UP.600 If $0<a\le b$ then:

$$a^3+3\int_a^b \sinh x\cdot \operatorname{arcsinh} x\,dx\ge b^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{For } all\ x \geq 0, \sinh x \overset{?}{\geq} x \Leftrightarrow P(x) = e^{2x} - 1 - 2xe^x \underset{(*)}{\underbrace{\overset{?}{\geq}}} 0$$

$$\text{Now, } P'(x) = 2e^x(e^x - x - 1) \geq 0 \Rightarrow P(x) \text{ is } \uparrow \text{ on } [0, \infty)$$

$$\Rightarrow P(x) \geq P(0) = 0 \Rightarrow \sinh x \overset{①}{\geq} x\ \forall\ x \geq 0$$

$$Also, \text{For } all\ x \geq 0, \cosh x \overset{?}{\geq} \sqrt{x^2+1} \Leftrightarrow Q(x) = \cosh x - \sqrt{x^2+1} \underset{(**)}{\underbrace{\overset{?}{\geq}}} 0$$

$$\text{Now, } Q'(x) = \sinh x - \frac{x}{\sqrt{x^2+1}} \overset{\text{via } ①}{\geq} x - \frac{x}{\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}.\left(\sqrt{x^2+1} - 1\right)$$

$$= \frac{x}{\sqrt{x^2+1}}.\frac{x^2}{\sqrt{x^2+1}+1} \overset{x \geq 0}{\geq} 0 \Rightarrow Q(x) \text{ is } \uparrow \text{ on } [0, \infty)$$

$$\Rightarrow Q(x) \geq Q(0) = 0 \Rightarrow \cosh x \overset{②}{\geq} \sqrt{x^2+1}\ \forall\ x \geq 0$$

$$Again, \text{For } all\ x \geq 0, arcsinh\, x \overset{?}{\geq} \frac{x}{(x^2+1)^{\frac{3}{2}}} \Leftrightarrow R(x) = arcsinh\, x - \frac{x}{(x^2+1)^{\frac{3}{2}}} \underset{(***)}{\underbrace{\overset{?}{\geq}}} 0$$

$$\text{Now, } R'(x) = \frac{1}{\sqrt{x^2+1}} - \frac{1}{(x^2+1)^{\frac{3}{2}}} + \frac{3x^2}{(x^2+1)^{\frac{5}{2}}} = \frac{x^4+4x^2}{(x^2+1)^{\frac{5}{2}}} \overset{x \geq 0}{\geq} 0$$

$$\Rightarrow R(x) \text{ is } \uparrow \text{ on } [0, \infty) \Rightarrow R(x) \geq R(0) = 0 \Rightarrow arcsinh\, x \overset{③}{\geq} \frac{x}{(x^2+1)^{\frac{3}{2}}}\ \forall\ x \geq 0$$

$$and\ now, let\ F(x) = \sinh x . arcsinh\, x - x^2\ \forall\ x \geq 0\ and\ then:$$

$$F'(x) = \frac{\sinh x}{\sqrt{x^2+1}} + (arcsinh\, x)(\cosh x) - 2x\ and$$

$$F''(x) = (\sinh x)\left(arcsinh\, x - \frac{x}{(x^2+1)^{\frac{3}{2}}}\right) + \frac{2}{\sqrt{x^2+1}}\left(\cosh x - \sqrt{x^2+1}\right)$$

$$\overset{\text{via } ② \text{ and } ③}{\geq} 0 \left(\because \sinh x \overset{\text{via } ①}{\geq} x \geq 0\right) \therefore F'(x) \text{ is } \uparrow \text{ on } [0, \infty) \Rightarrow F'(x) \geq F'(0) = 0$$

$$\Rightarrow F(x) \text{ is } \uparrow \text{ on } [0, \infty) \Rightarrow F(x) \geq F(0) = 0 \therefore \sinh x . arcsinh\, x \geq x^2\ \forall\ x \geq 0$$

$$and \because 0 < a \leq b \therefore \int_a^b \sinh x . arcsinh\, x\, dx \geq \int_a^b x^2 dx = \frac{1}{3}[x^3]_a^b = \frac{b^3 - a^3}{3}$$

$$\Rightarrow a^3 + 3\int_a^b \sinh x . arcsinh\, x\, dx \geq b^3$$

Solution 2 by proposer

Let be $f: [0, \infty) \to \mathbb{R}; f(x) = \sinh x - x$

$$f'(x) = \cosh x - 1$$

$$f'(x) = 0 \Rightarrow \cosh x - 1 = 0 \Rightarrow \cosh x = 1$$

$$\frac{e^x - e^{-x}}{2} - 1 = 0 \Rightarrow e^x - 2 + e^{-x} = 0 \Rightarrow \left(\sqrt{e^x} - \sqrt{e^{-x}}\right)^2 = 0$$

$$\sqrt{e^x} - \sqrt{e^{-x}} = 0 \Rightarrow e^x = e^{-x} \Rightarrow x = -x \Rightarrow 2x = 0 \Rightarrow x = 0$$

$$\min_{x \geq 0} f(x) = f(0) = 0 \Rightarrow f(x) \geq 0; (\forall) x \geq 0$$

$$\sinh x - x \geq 0 \Rightarrow \sinh x \geq x \Rightarrow \sinh^2 x \geq x^2$$

$$1 + \sinh^2 x \geq 1 + x^2 \Rightarrow \cosh^2 x \geq 1 + x^2 \Rightarrow$$

$$\sqrt[4]{\cosh^2 x} \geq \sqrt[4]{1 + x^2} \Rightarrow \sqrt{\cosh x} \geq \sqrt[4]{1 + x^2}$$

$$\frac{\sqrt{\cosh x}}{\sqrt[4]{1+x^2}} \geq 1 \qquad (1)$$

By Cauchy – Schwarz's inequality (integral form):

$$\int_0^x \left(\sqrt{\cosh x}\right)^2 dx \cdot \int_0^x \left(\frac{1}{\sqrt[4]{1 + x^2}}\right)^2 dx \geq \left(\int_0^x \frac{\sqrt{\cos x}}{\sqrt[4]{1 + x^2}} dx\right)^2 \overset{(1)}{\geq} \left(\int_0^x dx\right)^2 = x^2$$

$$\left(\int_0^x \cosh x\right)\left(\int_0^x \frac{1}{\sqrt{1 + x^2}} dx\right) \geq x^2$$

$$\sinh x \cdot \ln\left(x + \sqrt{1 + x^2}\right) \geq x^2$$

$$\sinh x \cdot \operatorname{arcsinh} x \geq x^2$$

$$\int_a^b (\sinh x \cdot \operatorname{arcsinh} x)\, dx \leq \int_a^b x^2\, dx = \frac{b^3 - a^3}{3}$$

$$3 \int_a^b (\sinh x \cdot \operatorname{arcsinh} x)\, dx \geq b^3 - a^3$$

$$a^3 + 3 \int_a^b (\sinh x \cdot \operatorname{arcsinh} x)\, dx \geq b^3$$

Equality holds for $a = b$.