Supraarithmetic means , subharmonic means

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In this article, *supraarithmetic* and sub*harmonic means* are introduced and studied. Some examples of such *means* are analyzed. The correlations between these types of means and the *mean's conjugation operator* are also highlighted, as well as the *refinement* relation that is established using these *means.*

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We recall that a **(*n-ary*) *mean*** in a set $\mathbf{S} \subset \mathbb{R}$ is a function $\mathbf{M} : \mathbf{S}^n \longrightarrow \mathbf{S}$, with the property of ***internality*** :

$$min\{a_1, a_2, \ldots, a_n\} \leq \mathbf{M}(a_1, a_2, \ldots, a_n) \leq max\{a_1, a_2, \ldots, a_n\}, \qquad \mathbf{(1)}$$

$(\forall)\ a_1, a_2, \ldots, a_n \in \mathbf{S}$.

In general, for the numbers $a_1, a_2, \ldots, a_n > 0$, $n \in \mathbb{N}^*$ we will denote and define their *classical means* , as follows , [1] :

$$A_n(a_1, a_2, \ldots, a_n) := \frac{a_1 + a_2 + \ldots + a_n}{n} , \quad (\textit{arithmetic mean}) \qquad \mathbf{(2)}$$

$$G_n(a_1, a_2, \ldots, a_n) := \sqrt[k]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} , \quad (\textit{geometric mean}) \qquad \mathbf{(3)}$$

$$H_n(a_1, a_2, \ldots, a_n) := \frac{n}{1/a_1 + 1/a_2 + \ldots + 1/a_n} , \quad (\textit{harmonic mean}) . \qquad \mathbf{(4)}$$

Obviously there are numerous other *means* , theoretically - infinitely many. Among them, we will be interested here (and we will introduce them in the following) in two categories of *means* , *namely supraarithmetic means* and *subharmonic means :*

1. *Definition*

Let the numbers $a_1, a_2, \ldots, a_n \in \mathbf{S} \subset \mathbb{R}$, $n \in \mathbb{N}^*$.

A *mean* $\mathcal{M}(a_1, a_2, \ldots, a_n)$ is called ***supraarithmetic mean*** if ,

$$\mathcal{M}(a_1, a_2, \ldots, a_n) \geq \mathrm{A}_n(a_1, a_2, \ldots, a_n) . \qquad \mathbf{(5)}$$

A *mean* $\mathcal{N}(a_1, a_2, \ldots, a_n)$ it is called ***subharmonic mean*** if,

$$\mathcal{N}(a_1, a_2, \dots, a_n) \leq \mathrm{H}_n(a_1, a_2, \dots, a_n) \ . \qquad (6)$$

2. *Remark*

When there is no possibility of confusion regarding the numbers $a_1, a_2, \dots, a$, then relations (5) and (6) are written in simplified form: $\mathcal{M} \geq \mathrm{A}_n$, $\mathcal{N} \leq \mathrm{H}_n$.

Furthermore, if we also note : $m := min\{a_1, a_2, \dots, a_n\}$, $M := max\{a_1, a_2, \dots, a_n\}$, then we can express the relations from the previous definition in the language of membership to intervals :

$\mathcal{M}$ is the ***supraarithmetic mean*** if, $\mathcal{M} \in [\mathrm{A}_n, \mathrm{M}]$ **(7)**

$\mathcal{N}$ is the ***subharmonic mean*** if, $\mathcal{N} \in [m, \mathrm{H}_n]$ (8)

Characteristic to many types of means are the properties (see e.g.: [1] , [4] , [5]) :

– *of* ***symmetry*** : if $\mathrm{M}(a, b) = \mathrm{M}(b, a)$, $(\forall)\, a, b \in \mathrm{S}$;

– *of* ***homogeneity*** : if $\mathrm{M}(ka, kb) = k\,\mathrm{M}(b, a)$, $(\forall)\, a, b \in \mathrm{S}$, $k \in \mathbb{R}_{>0}$.

3. *Remember*

If **M** and **N** there are two *binary means* , we say that **M** and **N** are *inverse* (with respect to **G**) if $\mathrm{M} \cdot \mathrm{N} = \mathrm{G}^2$, and we will say that **N** is the *inverse* of **M** and we will write $\mathrm{N} = {}^i\mathrm{M}$.

Of course, as well , **M** is the *inverse* of **N** , too and ${}^i\left({}^i\mathrm{M}\right) = \mathrm{M}$ ·

The notion of *inverse mean* was introduced by *Corrado Gini*, in [1] .See also [4] , [5] , [8].

For example - for *classical means* , the *arithmetic mean* and the *harmonic mean* are *inverse means* : ${}^iA = H$, ${}^iH = A$, as can be easily observed. The *geometric mean* is its own inverse , ${}^iG = G$. In [7], the notion of conjugation was also introduced .

4. *Definition*

For the mean $\mathcal{M}(a_1, a_2, \dots, a_n)$ of numbers $a_1, a_2, \dots, a_n \in \mathbb{R}_{>0}$ we will call its *conjugate* , the expression denoted and defined as follows:

$${}^c\mathcal{M}(a_1, a_2, \dots, a_n) := 1 / \mathcal{M}[1/a_1, 1/a_2, \dots 1/a_n] \ .$$

The expressions $\mathcal{M}(a_1, a_2, \dots, a_n)$ and ${}^c\mathcal{M}(a_1, a_2, \dots, a_n)$ will be called *conjugate* expressions .

Also, the *conjugation* operator is *involutive* : ${}^c\left({}^c\mathrm{M}\right) = \mathrm{M}$, as can be easily observed.

As well as *inverse means* , the *arithmetic mean* and the *harmonic mean* are *conjugate means* : ${}^cA = H$, ${}^cH = A$, as can be easily observed. The *geometric mean* is its own

There are situations when the *inversion* and *conjugation* operators are different, see for example in [4] .

For what follows, the following is very important ,

5. *Lemma*

For the numbers $a_1, a_2, \ldots, a_n \in \mathbb{R}_{>0}$, the following identities hold :

(a) $$\frac{1}{max(1/a_1, 1/a_2, \ldots, 1/a_n)} = min(a_1, a_2, \ldots, a_n) ; \qquad \textbf{(9)}$$

(b) $$\frac{1}{min(1/a_1, 1/a_2, \ldots, 1/a_n)} = max(a_1, a_2, \ldots, a_n). \qquad \textbf{(10)}$$

Proof

(a) We will prove by induction . For $n=2$, we have to prove that :

$$\frac{1}{max(1/a_1, 1/a_2)} = min(a_1, a_2) . \qquad \textbf{(9')}$$

Considering $a_1 \le a_2$, the relation **(9')** becomes , $\frac{1}{1/a_1} = a_1$, obviously true .

We assume that $\frac{1}{max(1/a_1, 1/a_2, \ldots, 1/a_{n-1})} = min(a_1, a_2, \ldots, a_{n-1})$; and we will demonstrate that $\frac{1}{max(1/a_1, 1/a_2, \ldots, 1/a_n)} = min(a_1, a_2, \ldots, a_n)$. Indeed, we have:

$$\frac{1}{max(1/a_1, 1/a_2, \ldots, 1/a_n)} = \frac{1}{max((1/a_1, 1/a_2, \ldots, 1/a_{n-1}), 1/a_n)} =$$
$$= min((a_1, a_2, \ldots, a_{n-1}), a_n) = min(a_1, a_2, \ldots, a_n) .$$

(b) In a similar way .

With the following result, we have a characterization of *supraarithmetic* and *subharmonic means* .

6. *Theorem*

If $a_1, a_2, \ldots, a_n \in \mathbb{R}_{>0}$, then occurs the statement :
$\mathcal{M}(a_1, a_2, \ldots, a_n)$ is a *supraarithmetic mean* if and only if ${}^c\mathcal{M}(a_1, a_2, \ldots, a_n)$ ts the *subharmonic mean* .

Proof

$\mathcal{M}(a_1, a_2, \ldots, a_n)$ is a *supraarithmetic mean* if and only if
$\mathcal{M}(a_1, a_2, \ldots, a_n) \ge A_n(a_1, a_2, \ldots, a_n)$, $(\forall)\, a_1, a_2, \ldots, a_n \in \mathbb{R}_{>0}$. In particular , making the substitutions : $a_1 \to 1/a_1$, $a_2 \to 1/a_2$, … , $a_n \to 1/a_n$, we obtain:

$$M[1/a_1,1/a_2,\dots,1/a_n] \geq A_n(1/a_1,1/a_2,\dots,1/a_n) \Leftrightarrow$$

$$\Leftrightarrow \quad 1/M[1/a_1,1/a_2,\dots,1/a_n] \leq 1/A_n(1/a_1,1/a_2,\dots,1/a_n) \Leftrightarrow$$

$$\Leftrightarrow \quad {}^{c}M[a_1,a_2,\dots,a_n] \leq H_n(a_1,a_2,\dots,a_n) \; .$$

Definition 3 was used , as well as the well-known identity :

$$1/A_n(1/a_1,1/a_2,\dots,1/a_n) = H_n(a_1,a_2,\dots,a_n) \; .$$

Several *supraarithmetic* and *subharmonic means* have been proposed in [3] .

7. Proposition , [3]

For any real numbers $a , b > 0$, the following inequalities hold :

(a)

$$\frac{a+b}{2} \overset{①}{\leq} \sqrt{\frac{a^2+ab+b^2}{3}} \overset{②}{\leq} \sqrt{\frac{a^2+b^2}{2}} \overset{③}{\leq} \frac{a^2+b^2}{a+b} \overset{④}{\leq} \sqrt{a^2-ab+b^2} \overset{⑤}{\leq} max\{a,b\} \; . \tag{11}$$

(inequalities for superarithmetic means)

(b)

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \overset{①}{\geq} \sqrt{\frac{3}{\frac{1}{a^2}+\frac{1}{ab}+\frac{1}{b^2}}} \overset{②}{\geq} \sqrt{\frac{2}{\frac{1}{a^2}+\frac{1}{b^2}}} \overset{③}{\geq} \frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}+\frac{1}{b^2}} \overset{④}{\geq} \frac{1}{\sqrt{\frac{1}{a^2}-\frac{1}{ab}+\frac{1}{b^2}}} \overset{⑤}{\geq} min\{a,b\} \; . \tag{12}$$

(inequalities for subharmonicc means)

Proof

(a) After routine calculations, the inequalities ①, ②, ③, ④ reduce to the obvious inequality, $a^2+b^2 \geq 2ab$.

For inequality ⑤ (due to symmetry in a , b) we assume $a \leq b$ and then the inequality becomes ,

$$\sqrt{a^2-ab+b^2} \leq b \quad \Leftrightarrow \quad a^2-ab+b^2 \leq b^2 \quad \Leftrightarrow \quad a^2 \leq ab \quad \Leftrightarrow \quad a \leq b \; .$$

In all inequalities, the equality relation occurs when $a = b$.

(b) The inequalities in (b) are obtained from (a) , based on Theorem 6 .

8. *Remark*

In *Proposition* 7 , besides the ***classical binary means***:

$$A_2(a,b) := \frac{a+b}{2} \quad , \quad (\textit{arithmetic mean}) \; ; \qquad (13)$$

$$H_2(a,b) := \frac{2}{1/a+1/b} \quad , (\textit{harmonic mean}) \; ; \qquad (14)$$

$$Q_2(a,b) := \sqrt{\frac{a^2+b^2}{2}} \quad , \quad (\textit{quadratic mean}) \; , \qquad (15)$$

other *means* also appear, which we will name and note in a *sui=generis* way :

$${}^cQ_2(a,b) := \sqrt{\frac{2}{\frac{1}{a^2}+\frac{1}{b^2}}} \quad , (c\text{−}\textit{quadratic mean}) \; , \qquad (16)$$

$$C_2(a,b) := \frac{a^2+b^2}{a+b} \quad , \quad (\textit{contraharmonic mean}) \; , \qquad (17)$$

$${}^cC_2(a,b) := \frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}+\frac{1}{b^2}} \quad , \quad (c\text{−}\textit{contraharmonic mean}) \; , \qquad (18)$$

$$P_2(a,b) := \sqrt{\frac{a^2+ab+b^2}{3}} \quad , \quad (\textit{extended power mean}) \; , \qquad (19)$$

$${}^cP_2(a,b) := \sqrt{\frac{3}{\frac{1}{a^2}+\frac{1}{ab}+\frac{1}{b^2}}} \quad , (c\text{−}\textit{extended power mean}) \qquad (20)$$

$$R_2(a,b) := \sqrt{a^2-ab+b^2} \quad , \quad (\textit{radical mean}) \; , \qquad (21)$$

$${}^cR_2(a,b) := \frac{1}{\sqrt{\frac{1}{a^2}-\frac{1}{ab}+\frac{1}{b^2}}} \quad , \quad (c\text{−}\textit{radical mean}) \qquad (22)$$

All the above *means* are *symmetric* and *homogeneous* . In addition, in each of the pairs : $\left(A_2(a,b), H_2(a,b)\right)$, $\left(Q_2(a,b), {}^cQ_2(a,b)\right)$, $\left(C_2(a,b), {}^cC_2(a,b)\right)$, $\left(P_2(a,b), {}^cP_2(a,b)\right)$, $\left(R_2(a,b), {}^cR_2(a,b)\right)$, $\left(max(a,b), min(a,b)\right)$, the first component is the *supraarithmetic mean*, and the second component is the *subharmonic mean* .

In [6] , the following multiple inequality was proposed :

- **If** $a, b > 0$, then :

$$\frac{a+b}{2} \le \ln \frac{e^a - e^b}{b-a} \le \ln \frac{e^a + e^b}{2} \le \frac{e^a + e^b - 2}{2} \quad . \qquad (23)$$

As we have , $\dfrac{e^a + e^b - 2}{2} \ge \max\{a, b\}$, **(24)**

we will retain from (23) only the first two inequalities :

9. *Proposition*

If $a, b > 0$, then we have the following inequalities :

(a)
$$\frac{a+b}{2} \overset{①}{\le} \ln \frac{e^b - e^a}{b-a} \overset{②}{\le} \ln \frac{e^a + e^b}{2} \overset{③}{\le} \max\{a, b\} \; ; \qquad (25)$$

(inequalities for supraarithmetic means)

(b)
$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \overset{①}{\ge} \frac{1}{\ln \frac{ab\left(e^{1/a} - e^{1/b}\right)}{b-a}} \overset{②}{\ge} \frac{1}{\ln \frac{e^{1/a} + e^{1/b}}{2}} \overset{③}{\ge} \min\{a, b\} \; . \qquad (26)$$

(inequalities for subharmonicc means)

Proof

(a) Applaying *Hermite-Hadamard inequality* for convex functions ,

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \cdot \int_a^b f(x) \cdot dx \le \frac{f(a)+f(b)}{2} \quad , \qquad (27)$$

for the function $f(x) = e^x$, we get :

$$e^{\frac{a+b}{2}} \le \frac{e^b - e^a}{b-a} \le \frac{e^a + e^b}{2} \quad .$$

By applying the logarithm , the inequalities ① , ② from the statement are obtained . For inequality ③, for $a \le b$, we have :

$$\ln \frac{e^a + e^b}{2} \le \ln \frac{2e^b}{2} \le \ln e^b = b = \max\{a, b\} \quad .$$

In all inequalities, the equality relation occurs when $a = b$.

(b) The inequalities in **(b)** are obtained from **(a)** , based on *Theorem* 6 .

10. ***Remark***

And here let us notice that in *Proposition* 9, besides the *classical binary means* : the *arithmetic mean* and the *harmonic mean* , new *binary means* also appear, which we will name and note as follows :

$$Lde_2(a,b) := \ln\frac{e^b - e^a}{b-a} \quad , \quad \textbf{(log. diff.exp. mean)} , \qquad \textbf{(28)}$$

$$^cLde_2(a,b) := \frac{1}{\ln\frac{ab\left(e^{1/a} - e^{1/b}\right)}{b-a}} \quad , \quad \textbf{(c– log. diff.exp. mean)} , \qquad \textbf{(29)}$$

$$Lse_2(a,b) := \ln\frac{e^a + e^b}{2} \quad , \quad \textbf{(log. sum exp. mean)} , \qquad \textbf{(30)}$$

$$^cLse_2(a,b) := \frac{1}{\ln\frac{e^{1/a} + e^{1/b}}{2}} \quad , \quad \textbf{(c– log. sum exp. mean)} . \qquad \textbf{(31)}$$

The means (28)-(31) are *symmetric* and *homogeneous* . In addition, in each of the pairs: $\left(A_2(a,b), H_2(a,b)\right)$, $\left(Lde_2(a,b), {}^cLde_2(a,b)\right)$, $\left(Lse_2(a,b), {}^cLse_2(a,b)\right)$, $\left(\max(a,b), \min(a,b)\right)$, the first component is the *supraarithmetic mean* , and the second component is the *subharmonic mean* .

In [5] , [6] , [7] – among others the following two inequalities were presented ;

$$\min(a,b) \le \left(a^b \cdot b^a\right)^{\frac{1}{a+b}} \le \frac{a+b}{2} \le \left(a^a \cdot b^b\right)^{\frac{1}{a+b}} \le \max(a,b) . \qquad \textbf{(32)}$$

$$\min(a,b,c) \le \left(a^{bc} \cdot b^{ca} \cdot c^{ab}\right)^{\frac{1}{ab+bc+ca}} \le \frac{a+b+c}{3} \le \left(a^a \cdot b^b \cdot c^c\right)^{\frac{1}{a+b+c}} \le \max(a,b,c) . \qquad \textbf{(33)}$$

In these inequalities we observe binary and trinary means which are *supraarithmetical neans* . We will try to generalize these *means* to the *n-ary case* .

11. ***Definition***

For the numbers $a_1, a_2, \ldots, a_n \in \mathbb{R}_{>0}$, we will consider the following two expressions :

$$E_n\left(a_1, a_2, \ldots, a_n\right) := \left(\prod_{k=1}^{n} a_k^{\,a_k}\right)^{\frac{1}{\sum_{k=1}^{n} a_k}} \quad , \qquad \textbf{(34)}$$

$$F_n\left(a_1,a_2,\ldots,a_n\right) := \left(\prod_{k=1}^{n} a_k^{\prod_{i=1,i\neq k}^{n} a_i}\right)^{\frac{1}{\sum_{k=1}^{n}\left(\prod_{i=1,i\neq k}^{n} a_i\right)}} . \quad (35)$$

12. *Proposition*

For any numbers $a_1, a_2, \ldots, a_n \in \mathbb{R}_{>0}$, the statements hold :

(a) The expression $E_n(a_1, a_2, \ldots, a_n)$ is *supraarithmetic mean* .

(b) The expression $F_n(a_1, a_2, \ldots, a_n)$ is *subharmonic mean* .

Proof

The expression $E_n(a_1, a_2, \ldots, a_n)$ is *supraarithmetic mean* of numbers $a_1, a_2, \ldots, a_n$, if we have , according to *Remark* 2 :

$$A_n\left(a_1,a_2,\ldots,a_n\right) \overset{①}{\leq} E_n\left(a_1,a_2,\ldots,a_n\right) \overset{②}{\leq} max\left\{a_1,a_2,\ldots,a_n\right\} . \quad (36)$$

Pentru inegalitatea ① , avem succesiv For inequality ① , we have successively :

$$A_n\left(a_1,a_2,\ldots,a_n\right) \leq E_n\left(a_1,a_2,\ldots,a_n\right) \Leftrightarrow ln\,A_n\left(a_1,a_2,\ldots,a_n\right) \leq ln\,E_n\left(a_1,a_2,\ldots,a_n\right) \Leftrightarrow$$

$$\Leftrightarrow ln\,A_n\left(a_1,a_2,\ldots,a_n\right) \leq \frac{1}{\sum_{k=1}^{n} a_k}\cdot\sum_{k=1}^{n} a_k\cdot ln\,a_k \Leftrightarrow \left(\sum_{k=1}^{n} a_k\right)\cdot ln\,A_n\left(a_1,a_2,\ldots,a_n\right) \leq \sum_{k=1}^{n} a_k\cdot ln\,a_k \Leftrightarrow$$

$$\Leftrightarrow n\cdot A_n\left(a_1,a_2,\ldots,a_n\right)\cdot ln\,A_n\left(a_1,a_2,\ldots,a_n\right) \leq \sum_{k=1}^{n} a_k\cdot ln\,a_k \Leftrightarrow$$

$$\Leftrightarrow A_n\left(a_1,a_2,\ldots,a_n\right)\cdot ln\,A_n\left(a_1,a_2,\ldots,a_n\right) \leq \frac{1}{n}\cdot\sum_{k=1}^{n} a_k\cdot ln\,a_k . \quad (*)$$

Using *Jensen's inequality* ,

$$f\left(\frac{a_1+a_2+\ldots+a_n}{n}\right) \leq \frac{f(a_1)+f(a_2)+\ldots+f(a_n)}{n} \Leftrightarrow$$

$$\Leftrightarrow f\left(A_n\left(a_1,a_2,\ldots,a_n\right)\right) \leq \frac{f(a_1)+f(a_2)+\ldots+f(a_n)}{n} , \quad (37)$$

for the convex *function* $f : (0,\infty) \longrightarrow \mathbb{R}$, $f(x) = x\cdot ln\,x$, we obtain :

$$A_n\left(a_1,a_2,\ldots,a_n\right)\cdot ln\left(A_n\left(a_1,a_2,\ldots,a_n\right)\right) \leq \frac{a_1\cdot ln\,a_1+a_2\cdot ln\,a_2+\ldots+a_n\cdot ln\,a_n}{n} ,$$

what is the inequality (*) above .

For the inequality ② , WLOG $a_1 \leq a_2 \leq \ldots \leq a_n$ and we will have :

$$E_n(a_1,a_2,\dots,a_n) := \left(\prod_{k=1}^{n} a_k^{a_k}\right)^{\frac{1}{\sum_{k=1}^{n} a_k}} \le \left(\prod_{k=1}^{n} a_n^{a_k}\right)^{\frac{1}{\sum_{k=1}^{n} a_k}} =$$

$$= \left(a_n^{\sum_{k=1}^{n} a_k}\right)^{\frac{1}{\sum_{k=1}^{n} a_k}} = a_n = max\{a_1,a_2,\dots,a_n\} \quad .$$

(b) We will demonstrate that $F_n(a_1, a_2, \dots, a_n)$ is the *conjugate* of the *mean* $E_n(a_1, a_2, \dots, a_n)$. Indeed, by *Definition* 4, we have :

$${}^{c}E_n(a_1,a_2,\dots,a_n) \overset{Def.4}{=} \frac{1}{E_n\left(\frac{1}{a_1},\frac{1}{a_2},\dots,\frac{1}{a_n}\right)} = \frac{1}{\left(\prod_{k=1}^{n}\left(\frac{1}{a_k}\right)^{\frac{1}{a_k}}\right)^{\frac{1}{\sum_{k=1}^{n}\frac{1}{a_k}}}} =$$

$$= \frac{1}{\dfrac{1}{\left(\prod_{k=1}^{n} a_k^{\frac{1}{a_k}}\right)^{\frac{\prod_{k=1}^{n} a_k}{\sum_{k=1}^{n}\left(\prod_{i=1,i\neq k}^{n} a_i\right)}}}} = \left(\prod_{k=1}^{n} a_k^{\frac{1}{a_k}}\right)^{\frac{\prod_{k=1}^{n} a_k}{\sum_{k=1}^{n}\left(\prod_{i=1,i\neq k}^{n} a_i\right)}} = \prod_{k=1}^{n} a_k^{\frac{1}{a_k}\cdot\frac{\prod_{k=1}^{n} a_k}{\sum_{k=1}^{n}\left(\prod_{i=1,i\neq k}^{n} a_i\right)}} =$$

$$= \prod_{k=1}^{n} a_k^{\frac{\prod_{i=1,i\neq k}^{n} a_k}{\sum_{k=1}^{n}\left(\prod_{i=1,i\neq k}^{n} a_i\right)}} = \left(\prod_{k=1}^{n} a_k^{\prod_{i=1,i\neq k}^{n} a_i}\right)^{\frac{1}{\sum_{k=1}^{n}\left(\prod_{i=1,i\neq k}^{n} a_i\right)}} = F_n(a_1,a_2,\dots,a_n) \quad .$$

The rest follows from *Theorem* 6.

Regarding the *supraarithmetic* and *subharmonic means* presented in this material it would be interesting to compare them and then establish an ordering of them. We will deal with this in a future paper .

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