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RIGOROUS CLOSURE OF KNEBELMAN'S DERIVATIVE OF THE SINE FUNCTION: FROM DISCRETE POLYGONS TO CONTINUOUS ANALYSIS

MARTIN CELLI

ABSTRACT. This note presents a complete derivation of the sine function at 0 by refining Knebelman's polygonal approximation.

A nice computation of the derivative of the sine function at 0

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1,$$

approximating a circle by regular polygons with an increasing number of sides, can be found in [1]. Unfortunately, it has not been as widely adopted as the classic computation based on the squeeze theorem. This may be due to two shortcomings.

- Only angles of the form $x = \pi/N$, where the integer N is the number of sides of the polygon, are taken into account. This gap is pointed out in [2].
- The perimeter of the polygons is not proved to tend to the circumference.

These difficulties are recurrent in mathematical analysis. While a complete proof may be within the reach of specialists, it probably deserves to be brought to the attention of non-specialists, not only to confirm Knebelman's argument, but also for its high educational value. This note aims to present such a proof.

In contrast to Knebelman's totally regular polygon, the polygon $P_0P_1P_2\dots P_N$ constructed here is composed of

- $N = [\pi/x]$ chords of angle $2x$ and length $2\sin(x)$, where $[\cdot]$ denotes the greatest integer function,
- one residual chord of angle $2\pi - [\pi/x]\cdot 2x$ and length $2\sin(\pi - [\pi/x]x)$.

The polygon has perimeter

$$2\left[\frac{\pi}{x}\right]\sin(x) + 2\sin\left(\pi - \left[\frac{\pi}{x}\right]x\right)\cdot$$

We have

$$0 \le \pi - \left[\frac{\pi}{x}\right]x < x,$$

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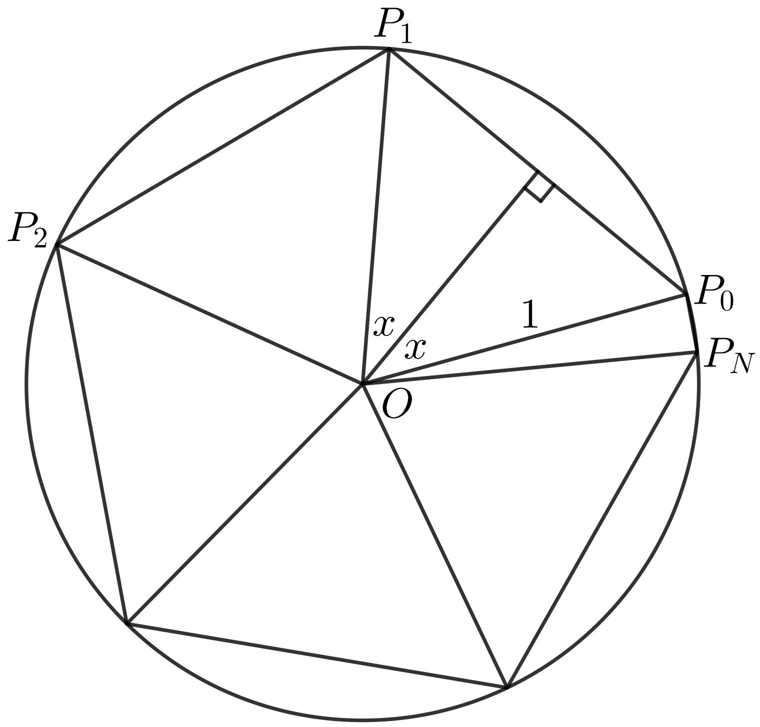

$$2\sin\left(\pi - \left[\frac{\pi}{x}\right] x\right) < 2\sin(x) \ll 2\left[\frac{\pi}{x}\right]\sin(x)\cdot$$

Thus, the perimeter is asymptotically equivalent to its first term

$$2\left[\frac{\pi}{x}\right]\sin(x) \sim 2\pi\frac{\sin(x)}{x}\cdot$$

To obtain the desired limit, it is sufficient to see that the perimeter also approaches the circumference 2π when $x \to 0$. This step, which was missing from [1], follows from recognizing the circumference as the line integral of the constant function 1 along the circle. The perimeter of every polygon then represents a corresponding Riemann sum whose mesh is the maximum chord length $2\sin(x)$ and approaches 0.

References

[1] M. S. Knebelman. *An elementary limit*. Am. Math. Month., 50, N. 8 (1943), p. 507. http://www.jstor.org/stable/2304191

[2] A. Bogomolny. *Why $sin(x)/x$ tends to* 1. Interactive Mathematics Miscellany and Puzzles (Cut The Knot). https://www.cut-the-knot.org/arithmetic/algebra/ElementaryLimit.shtml

DEPARTMENT OF MATHEMATICS, UNIVERSIDAD AUTÓNOMA METROPOLITANA
AV. FERROCARRIL SAN RAFAEL ATLIXCO, 186
C.P. 09310, COL. LEYES DE REFORMA 1A SECCIÓN
ALC. IZTAPALAPA, MEXICO CITY, MEXICO
Email address: `cell@xanum.uam.mx`