ON A CLASS OF POLYNOMIAL IDENTITIES

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We first consider a small and nice problem:

Problem 1: [Dang Ngoc Minh] Let three complex numbers x,y,z satisfy:

$x^2 + y^2 + z^2 + xy + yz + zx = 0$, then we have identity:

$$3(x^9 + y^9 + z^9 - (x + y + z)^9)^2 = (x^6 + y^6 + z^6 + (x + y + z)^6)^3$$

We can prove this problem by using the Girard–Newton identities. However, it is a consequence of another identity that I will present below:

*Let $x_1, ..., x_n$ be variables, denote for $k \geq 1$ by $p_k(x_1, ..., x_n)$ the k-th power sum:

$$p_k(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} x_i^k = x_1^k + x_2^k + ... + x_n^k$$

and for $k \geq 0$ denote by $e_k(x_1, ..., x_n)$ the elementary symmetric polynomial (that is, the sum of all distinct products of k distinct variables), so

$$e_0(x_1, x_2, ..., x_n) = 1$$

$$e_1(x_1, x_2, ..., x_n) = x_1 + x_2 + ... + x_n$$

$$e_2(x_1, x_2, ..., x_n) = \sum_{1 \leq i \leq j \leq n}^{n} x_i x_j$$

…

$$e_n(x_1, x_2, ..., x_n) = x_1 x_2 ... x_n$$

$$e_k(x_1, x_2,..., x_n) = 0 \; for \; k > n$$

and $N_2 = e_1^2 - e_2$; $N_k = \frac{p_k + (-e_1)^k}{k}$, $k \geq 3$. The following identities hold

$$e_1e_2 - e_3 = - N_3$$

$$e_1e_3 - e_4 = N_4 - \frac{1}{2}N_2^2$$

$$e_1e_4 - e_5 = - N_5 + N_2N_3$$

$$e_1e_5 - e_6 = N_6 - N_2N_4 - \frac{1}{2}N_3N_3 + \frac{1}{6}N_2^3$$

$$e_1e_6 - e_7 = - N_7 + \frac{1}{2}N_2N_5 + N_3N_4$$

$$e_1e_7 - e_8 = N_8 - \frac{1}{2}N_2N_6 - \frac{3}{4}N_3N_5 - \frac{1}{2}N_4N_4 + \frac{1}{24}N_2^4$$

$$e_1e_8 - e_9 = - N_9 + \frac{2}{9}N_2N_7 + \frac{2}{3}N_3N_6 + \frac{10}{9}N_4N_5$$

$$e_1e_9 - e_{10} = N_{10} - \frac{2}{7}N_2N_8 - \frac{4}{7}N_3N_7 - \frac{6}{7}N_4N_6 - \frac{3}{7}N_5N_5 + \frac{1}{70}N_2^5$$

$$e_1e_{10} - e_{11} = - N_{11} + \frac{3}{34}N_2N_9 + \frac{6}{17}N_3N_8 + \frac{14}{17}N_4N_7 + \frac{21}{17}N_5N_6$$

*For three variables (n=3), $N_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1$

and $N_k = \frac{x_1^k + x_2^k + x_3^k + (-1)^k (x_1 + x_2 + x_3)^k}{k}$, $for \; k \geq 3$, we obtain the following more compact identities:

$$N_5 = N_2N_3$$

$$N_6 = N_2N_4 + \frac{1}{2}N_3N_3 - \frac{1}{6}N_2^3$$

$N_7 = \frac{1}{2}N_2N_5 + N_3N_4 = \frac{1}{2}N_2^2N_3 + N_3N_4$

$N_8 = \frac{1}{2}N_2N_6 + \frac{3}{4}N_3N_5 + \frac{1}{2}N_4N_4 - \frac{1}{24}N_2^4 = N_2N_3^2 + \frac{1}{2}N_4^2 + \frac{1}{2}N_4N_2^2 - \frac{1}{8}N_2^4$

$N_9 = \frac{2}{9}N_2N_7 + \frac{2}{3}N_3N_6 + \frac{10}{9}N_4N_5 = \frac{1}{3}N_3^3 + 2N_2N_3N_4$

$N_{10} = \frac{2}{7}N_2N_8 + \frac{4}{7}N_3N_7 + \frac{6}{7}N_4N_6 + \frac{3}{7}N_5N_5 - \frac{1}{70}N_2^5 = N_4N_3^2 + N_2^2N_3^2 + N_2N_4^2 - \frac{1}{20}N_2^5$

$N_{11} = \frac{3}{34}N_2N_9 + \frac{6}{17}N_3N_8 + \frac{14}{17}N_4N_7 + \frac{21}{17}N_5N_6 = N_2N_3^3 + 2N_2^2N_3N_4 + N_3N_4^2 - \frac{1}{4}N_3N_2^4$

Some problems can be proved using the above identities

<u>Problem 2</u>: If x+y+z=0 then prove that:

$$\left(\frac{x^5+y^5+z^5}{5}\right)^2 = \frac{x^3+y^3+z^3}{3}.\frac{x^7+y^7+z^7}{7}$$

<u>Problem 3</u>: If x+y+z=0 then prove that:

$$16(x^3 + y^3 + z^3)(x^5 + y^5 + z^5) + 15(x^4 + y^4 + z^4)^2 = 30(x^8 + y^8 + z^8)$$

<u>Problem 4</u>: If x+y+z=0 then prove that:

$$3(x^2 + y^2 + z^2)^3 + 4(x^3 + y^3 + z^3)^2 = 12(x^6 + y^6 + z^6)$$

<u>Problem 5</u>: For x,y,z are complex numbers, such that:

$x^2 + y^2 + z^2 + xy + yz + zx = 0$, prove that:

$$(x^2 + y^2 + z^2)^5 + (x^5 + y^5 + z^5)^2 = 0$$

Problem 6: For x,y,z are real numbers, prove that:

$$6(x^5 + y^5 + z^5) - 5(x^3 + y^3 + z^3)(x^2 + y^2 + z^2)$$

$$= (x + y + z)^2[(x + y + z)^3 - 5(x + y + z)(xy + yz + zx) + 15xyz]$$

Problem 7: [Lame] For x,y,z are real numbers, prove that:

$(x+y+z)^7 - (x^7+y^7+z^7) = 7(x+y)(x+z)(y+z)[(x^2+y^2+z^2+xy+xz+yz)^2+xyz(x+y+z)]$

Problem 8: [Candido] For x,y are real numbers, prove that:

$$2[x^4 + y^4 + (x + y)^4] = [x^2 + y^2 + (x + y)^2]^2$$

Problem 9: Show that the following rational function reduces to a polynomial:

$$\frac{(x+y+z)(x^7+y^7+z^7)-(x^3+y^3+z^3)(x^5+y^5+z^5)}{(x+y)(y+z)(z+x)}$$

Proof:

$$(x + y + z)(x^7 + y^7 + z^7) - (x^3 + y^3 + z^3)(x^5 + y^5 + z^5)$$

$$= p_1p_7 - p_3p_5 = p_1(7N_7 + p_1^7) - (3N_3 + p_1^3)(5N_5 + p_1^5)$$

$$= 7p_1N_7 - 3N_3p_5 - 5p_1^3N_5$$

$$= 7p_1(\tfrac{1}{2}N_2^2N_3 + N_3N_4) - 3N_3p_5 - 5p_1^3N_2N_3$$

$$= N_3[7p_1(\tfrac{1}{2}N_2^2 + N_4) - 3p_5 - 5p_1^3N_2]$$

$$=- (x + y)(y + z)(z + x)[7p_1(\tfrac{1}{2}N_2^2 + N_4) - 3p_5 - 5p_1^3N_2]$$

The proof is complete.

Problem 10: [Ramanujan]. For a,b,c,d are real numbers such that: ad=bc, prove that:

$$64[(a+b+c)^6+(b+c+d)^6+(a-d)^6-(c+d+a)^6-(d+a+b)^6-(b-c)^6][(a+b+c)^{10}+(b+c+d)^{10}+(a-d)^{10}-(c+d+a)^{10}-(d+a+b)^{10}-(b-c)^{10}] =$$

$$45[(a+b+c)^8+(b+c+d)^8+(a-d)^8-(c+d+a)^8-(d+a+b)^8-(b-c)^8]^2$$

Proof: Let : $a_1 = a + b + c$, $a_2 =- b - c - d,\ a_3 =- a + d$

$b_1 =- c - d - a$, $b_2 = d + a + b,\ b_3 =- b + c$

$F_n = \frac{1}{n}(a_1^n + a_2^n + a_3^n - b_1^n - b_2^n - b_3^n)$

$a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 0$

Because ad=bc then $F_2 = 6(bc - ad) = 0$

We need to show: $3F_8^2 = 4F_6F_{10}$

Again, let: $N_n = \frac{1}{n}(a_1^n + a_2^n + a_3^n + (- a_1 - a_2 - a_3)^n)$

$M_n = \frac{1}{n}(b_1^n + b_2^n + b_3^n + (- b_1 - b_2 - b_3)^n)$

$F_n = N_n - M_n$

with $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 0$ then $N_2^2 = 2N_4$ and $M_2^2 = 2M_4$

easy to see that: $F_2 = 0\ then\ N_2 = M_2\ and\ N_4 = M_4$

$$F_6 = N_6 - M_6 = (N_2N_4 + \frac{1}{2}N_3^{\ 2} - \frac{1}{6}N_2^{\ 3}) - (M_2M_4 + \frac{1}{2}M_3^{\ 2} - \frac{1}{6}M_2^{\ 3})$$

$$F_6 = \frac{1}{2}(N_3^{\ 2} - M_3^{\ 2})$$

$$F_8 = N_8 - M_8 = (N_2N_3^{\ 2} + \frac{1}{2}N_4^{\ 2} + \frac{1}{2}N_4N_2^{\ 2} - \frac{1}{8}N_2^{\ 4}) - (M_2M_3^{\ 2} + \frac{1}{2}M_4^{\ 2} + \frac{1}{2}M_4M_2^{\ 2} - \frac{1}{8}M_2^{\ 4})$$

$$F_8 = N_2(N_3^{\ 2} - M_3^{\ 2})$$

$$F_{10} = N_{10} - M_{10} = (N_4N_3^{\ 2} + N_2^{\ 2}N_3^{\ 2} + N_2N_4^{\ 2} - \frac{1}{20}N_2^{\ 5}) - (M_4M_3^{\ 2} + M_2^{\ 2}M_3^{\ 2} + M_2M_4^{\ 2} - \frac{1}{20}M_2^{\ 5})$$

$$F_{10} = \frac{3}{2}N_2^{\ 2}(N_3^{\ 2} - M_3^{\ 2})$$

$$4F_6F_{10} = 4.\frac{1}{2}(N_3^{\ 2} - M_3^{\ 2})\frac{3}{2}N_2^{\ 2}(N_3^{\ 2} - M_3^{\ 2}) = 3N_2^{\ 2}(N_3^{\ 2} - M_3^{\ 2})^2 = 3F_8^{\ 2}$$

The proof is complete.

Similarly, we can prove the other similar equalities

$$F_3F_7 = F_5^{\ 2}\ (Hirschhorn)$$

$$F_3F_8 = 2F_5F_6\ (Chamberland)$$

$$3F_6F_7 = 2F_3F_{10}\ (Chamberland)$$

$$F_3F_{11} + 2F_7^{\ 2} = 3F_5F_9\ (Chamberland)$$

$$9F_6F_{14} + 11F_{10}^{\ 2} = 18F_8F_{12}\ (Chamberland)$$

$$F_5F_8 = 2F_6F_7$$

$$2F_5F_{10} = 3F_7F_8$$