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NEW IDENTITIES WITH SPIEKER'S CEVIANS

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ABSTRACT: I this paper are proved new identities for Spieker's cevians in triangle.

We start from formula (22) in the article[1], for the Spieker's cevian from vertex A:

$$p_a = \frac{2p}{2p+a}\sqrt{4m_a^2 + a^2 - 2bc + 8Rr + 3r^2 - p^2}.$$

REDUCTION OF THE EXPRESSION UNDER THE SQUARE ROOT

We denote the radical expression by E: $\mathbf{E = 4m_a^2 + a^2 - 2bc + 8Rr + 3r^2 - p^2}$.

STEP 1: We use the identity for the median:

$$\mathbf{4m_a^2 = 2b^2 + 2c^2 - a^2. \text{ Hence: } E = 2(b^2 + c^2) - 2bc + 8Rr + 3r^2 - p^2.}$$

STEP 2: We use the classical identity:

$$\mathbf{a^2 + b^2 + c^2 = 2p^2 - 8Rr - 2r^2. \text{ Therefore: } b^2 + c^2 = 2p^2 - 8Rr - 2r^2 - a^2.}$$

$$\mathbf{\text{Substituting, we obtain: } E = 3p^2 - 8Rr - r^2 - 2a^2 - 2bc.}$$

STEP 3: We regroup terms and get: $\mathbf{E = p^2 + r^2 - (a^2 - (b-c)^2)}$.

STEP 4: Factorize the difference of squares: $\mathbf{a^2 - (b-c)^2 = (a-b+c)(a+b-c)}$.

$$\mathbf{\text{But: } a - b + c = 2(p-b), \qquad a + b - c = 2(p-c).}$$

$$\mathbf{\text{Thus: } a^2 - (b-c)^2 = 4(p-b)(p-c).}$$

RESULT: $\mathbf{E = p^2 + r^2 - 4(p-b)(p-c)}$.

FINAL FORM: Substituting back into formula (22):

$$\mathbf{p_a = \frac{2p}{2p+a}\sqrt{p^2 + r^2 - 4(p-b)(p-c)}.}$$

EXPRESSION ONLY IN TERMS OF $\mathbf{p, (p-a), (p-b), (p-c)}$

Using Heron's relation $r^2 = \frac{(p-a)(p-b)(p-c)}{p}$, we obtain:

$$\mathbf{p_a = \frac{2p}{2p+a}\sqrt{p^2 - 4(p-b)(p-c) + \frac{(p-a)(p-b)(p-c)}{p}}.(1)}$$

Useful results: $\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r}$(and analogous)[2]$\to \frac{h_a}{r} = \frac{2n_a}{\sqrt{4r^2+(b-c)^2}}$

$2S = ah_a = 2pr \to \frac{h_a}{r} = \frac{2p}{a} = 1 + \frac{b+c}{a}$

$\to \frac{2p}{a} = \frac{2n_a}{\sqrt{4r^2+(b-c)^2}}$; $\frac{2p+a}{2p} = 1 + \frac{\sqrt{4r^2+(b-c)^2}}{2n_a} = \frac{2n_a+\sqrt{4r^2+(b-c)^2}}{2n_a}$

From $\frac{2p+a}{2p} = \frac{2n_a+\sqrt{4r^2+(b-c)^2}}{2n_a}$ and (1) , obtain:

$$\mathbf{p_a = \frac{2n_a}{2n_a+\sqrt{4r^2+(b-c)^2}}\sqrt{p^2 - 4(p-b)(p-c) + \frac{(p-a)(p-b)(p-c)}{p}}(2)}$$

Useful results :$(p-b)(p-c) = r_a r$

$$\mathbf{p_a = \frac{2n_a}{2n_a+\sqrt{4r^2+(b-c)^2}}\sqrt{p^2 - 4r_a r + r^2}(3)}$$

Useful result :$\frac{r_a}{r} = \frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}$ (and analogous)[3] and use (3):

$$\mathbf{\frac{p_a}{r} = \frac{2n_a}{2n_a+\sqrt{4r^2+(b-c)^2}}\sqrt{\left(\frac{p}{r}\right)^2 - \frac{4n_a}{n_a-\sqrt{4r^2+(b-c)^2}} + 1}\ (4)}$$

$$\mathbf{\frac{2n_a+\sqrt{4r^2+(b-c)^2}}{2\sqrt{p^2-4r_a r+r^2}}\frac{p_a}{h_a} = \frac{n_a}{h_a}\ (5)}$$

Useful result: $\frac{n_a}{h_a}, \frac{n_b}{h_b}, \frac{n_c}{h_c}$ can be sides of a triangle[2],and from (5):

$\mathbf{\frac{2n_a+\sqrt{4r^2+(b-c)^2}}{2\sqrt{p^2-4r_a r+r^2}}\frac{p_a}{h_a}, \frac{2n_b+\sqrt{4r^2+(a-c)^2}}{2\sqrt{p^2-4r_b r+r^2}}\frac{p_b}{h_b}, \frac{2n_c+\sqrt{4r^2+(b-a)^2}}{2\sqrt{p^2-4r_c r+r^2}}\frac{p_c}{h_c}}$ **can be sides of a triangle (6)**

Useful result:$\frac{n_a}{r_a} = \frac{n_a-\sqrt{4r^2+(b-c)^2}}{r}$ (and analogous)[3] and (3):

$\frac{p_a}{r_a} = \frac{2n_a}{r_a}\frac{\sqrt{p^2-4r_a r+r^2}}{2n_a+\sqrt{4r^2+(b-c)^2}} \to \frac{p_a}{r_a} = \frac{2\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}{r}\frac{\sqrt{p^2-4r_a r+r^2}}{2n_a+\sqrt{4r^2+(b-c)^2}}$

$$\mathbf{\frac{p_a}{r_a} = \frac{2\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}{2n_a+\sqrt{4r^2+(b-c)^2}}\sqrt{\left(\frac{p}{r}\right)^2 - \frac{4n_a}{n_a-\sqrt{4r^2+(b-c)^2}} + 1}\ (7)}$$

Useful identity: $a = \sqrt{(r_b + r_c)(r_a - r)}$ (and analogous);

$4R+r=r_a+r_b+r_c \rightarrow a=\sqrt{(4R-(r_a-r))(r_a-r)}$

$\frac{a}{r}=\frac{\sqrt{(4R-(r_a-r))(r_a-r)}}{r}$ and $\frac{r_a}{r}=\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}$ (and analogous)

$$\frac{\mathbf{a}}{\mathbf{r}}=\sqrt{\left(\frac{\mathbf{4R}}{\mathbf{r}}-\left(\frac{\mathbf{n_a}}{\mathbf{n_a}-\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}\right)\right)\left(\frac{\mathbf{n_a}}{\mathbf{n_a}-\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}\right)}\text{(and analogous) (8)}$$

$$\sqrt{\left(\frac{\mathbf{4R}}{\mathbf{r}}-\left(\frac{\mathbf{n_a}}{\mathbf{n_a}-\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}\right)\right)\left(\frac{\mathbf{n_a}}{\mathbf{n_a}-\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}\right)},$$
$$\sqrt{\left(\frac{\mathbf{4R}}{\mathbf{r}}-\left(\frac{\mathbf{n_b}}{\mathbf{n_b}-\sqrt{\mathbf{4r^2+(a-c)^2}}}-\mathbf{1}\right)\right)\left(\frac{\mathbf{n_b}}{\mathbf{n_b}-\sqrt{\mathbf{4r^2+(a-c)^2}}}-\mathbf{1}\right)},$$
$$\sqrt{\left(\frac{\mathbf{4R}}{\mathbf{r}}-\left(\frac{\mathbf{n_c}}{\mathbf{n_c}-\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}\right)\right)\left(\frac{\mathbf{n_c}}{\mathbf{n_c}-\sqrt{\mathbf{4r^2+(b-a)^2}}}-\mathbf{1}\right)}$$

-can be sides of a triangle(9)

Useful result:$n_a \geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogous)[4] and from (8), obtain:

$$\frac{\mathbf{a}}{\mathbf{r}}\leq\sqrt{\left(\frac{\mathbf{4R}}{\mathbf{r}}-\left(\frac{\mathbf{n_a}}{\mathbf{p_a}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}}-\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}\right)\right)\left(\frac{\mathbf{n_a}}{\mathbf{p_a}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}}-\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}\right)}\text{ (and analogous) (10)}$$

From (10) after summation:

$$\frac{\mathbf{2p}}{\mathbf{r}}\leq\sum\sqrt{\left(\frac{\mathbf{4R}}{\mathbf{r}}-\left(\frac{\mathbf{n_a}}{\mathbf{p_a}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}}-\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}\right)\right)\left(\frac{\mathbf{n_a}}{\mathbf{p_a}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}}-\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}\right)}\text{ (11)}$$

Useful result:$4m_a^2=n_a^2+g_a^2+2r_br_c$ (and analogous)[5] and using means inequality:

$\sqrt{\frac{n_a^2+g_a^2+2r_br_c}{4}}\geq\frac{n_a+g_a+2\sqrt{r_br_c}}{4}\longrightarrow\frac{2m_a}{2}\geq\frac{n_a+g_a+2\sqrt{r_br_c}}{4}$

$$\mathbf{4m_a \geq n_a+g_a+2\sqrt{r_br_c}}\text{ (and analogous) (12)}$$

$4m_a - n_a - g_a \geq 2\sqrt{r_br_c}$;

Useful result: $2r_br_c = h_a(r_b + r_c)$(and analogous);

$$\frac{h_a}{r} = \frac{2n_a}{\sqrt{4r^2+(b-c)^2}};\ \frac{r_b+r_c}{r} = \frac{4R}{r} - \left(\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}} - 1\right);$$

$$\frac{2r_br_c}{r^2} = \frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\left[\frac{4R}{r} - \left(\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}} - 1\right)\right]$$

$$\mathbf{\frac{r_br_c}{r^2} = \frac{n_a}{\sqrt{4r^2+(b-c)^2}}\left[\frac{4R}{r} - \left(\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}} - 1\right)\right]}\textbf{(and analogous)(13)}$$

$$\frac{\sqrt{r_br_c}}{r} = \frac{\sqrt{n_a}}{\sqrt[4]{4r^2+(b-c)^2}}\sqrt{\left[\frac{4R}{r} - \left(\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}} - 1\right)\right]}$$

$4m_a - n_a - g_a \ge 2\sqrt{r_br_c}$;

$$\rightarrow \mathbf{\frac{4m_a - n_a - g_a}{2r} \ge \frac{\sqrt{n_a}}{\sqrt[4]{4r^2+(b-c)^2}}\sqrt{\left[\frac{4R}{r} - \left(\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}} - 1\right)\right]}}\textbf{(and analogous)(14)}$$

$$\rightarrow \mathbf{\frac{\sqrt[4]{4r^2+(b-c)^2}}{2r} \ge \frac{\sqrt{n_a}}{4m_a - n_a - g_a}\sqrt{\left[\frac{4R}{r} - \left(\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}} - 1\right)\right]}}\textbf{(and analogous)(15)}$$

$\frac{r_a}{r} = \frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}$; $4m_a - 2\sqrt{r_br_c} - g_a \ge n_a$

$4m_a - 2\sqrt{r_br_c} - g_a - \sqrt{4r^2 + (b-c)^2} \ge n_a - \sqrt{4r^2 + (b-c)^2}$

$$\mathbf{\frac{r_a}{r} \ge \frac{n_a}{4m_a - 2\sqrt{r_br_c} - g_a - \sqrt{4r^2+(b-c)^2}}}\ \textbf{(and analogous) (16)}$$

$$\mathbf{1 + \frac{4R}{r} \ge \sum \frac{n_a}{4m_a - 2\sqrt{r_br_c} - g_a - \sqrt{4r^2+(b-c)^2}}}\ \textbf{(17)}$$

$p_a = \frac{2n_a}{2n_a+\sqrt{4r^2+(b-c)^2}}\sqrt{p^2 - 4r_ar + r^2}$ and $4m_a - 2\sqrt{r_br_c} - g_a \ge n_a$

$$\rightarrow \frac{2(4m_a - 2\sqrt{r_br_c} - g_a)}{2n_a+\sqrt{4r^2+(b-c)^2}}\sqrt{p^2 - 4r_ar + r^2} \ge p_a$$

$$\rightarrow \mathbf{\frac{2\sqrt{p^2 - 4r_ar + r^2}}{2n_a+\sqrt{4r^2+(b-c)^2}} \ge \frac{p_a}{4m_a - 2\sqrt{r_br_c} - g_a}}\ \textbf{(and analogous)(18)}$$

From (14) and $n_a \ge p_a\sqrt{\frac{l_a}{g_a}}$ (and analogous):

$$\mathbf{\frac{4m_a - p_a\sqrt{\frac{l_a}{g_a}} - g_a}{2r} \ge \frac{\sqrt{n_a}}{\sqrt[4]{4r^2+(b-c)^2}}\sqrt{\left[\frac{4R}{r} - \left(\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}} - 1\right)\right]}}\textbf{(and analogous)(19)}$$

Useful result: $n_a + g_a \ge 2m_a$ (and analogous)[5] and $4m_a - n_a - g_a \ge 2\sqrt{r_br_c}$;

$$\mathbf{2m_a \geq 4m_a - n_a - g_a \geq 2\sqrt{r_b r_c}}\ \textbf{(and analogous)(20)}$$

Useful result: $\frac{b+c}{a}=1+\frac{h_a}{r_a}$; $\frac{h_a}{r}=\frac{2p}{a}=1+\frac{b+c}{a}\to\frac{h_a}{r}=2+\frac{h_a}{r_a}\to r_a h_a=(2r_a+h_a)r$

$p^2=n_a^2+2r_a h_a$ (and analogous)[6] ;$p^2=n_a^2+2r(2r_a+h_a)=n_a^2+4r_a r+2rh_a$ and (3)

$p_a=\frac{2n_a}{2n_a+\sqrt{4r^2+(b-c)^2}}\sqrt{p^2-4r_a r+r^2}=\frac{2n_a}{2n_a+\sqrt{4r^2+(b-c)^2}}\sqrt{n_a^2+4r_a r+2rh_a-4r_a r+r^2}=$
$\frac{2n_a}{2n_a+\sqrt{4r^2+(b-c)^2}}\sqrt{n_a^2+2rh_a+r^2}$;

$$\mathbf{p_a=\frac{2n_a}{2n_a+\sqrt{4r^2+(b-c)^2}}\sqrt{n_a^2+2rh_a+r^2}}\ \textbf{(and analogous)(21)}$$

From [1] :$p_a=\frac{4p}{2p+a}AS_p$ and $p_a=\frac{2p}{2p+a}\sqrt{p^2+r^2-4(p-b)(p-c)}$.

$$\mathbf{AS_p=\frac{1}{2}\sqrt{p^2+r^2-4(p-b)(p-c)}}\ \textbf{(and analogous) (22)}$$

$$\mathbf{AS_p=\frac{1}{2}\sqrt{n_a^2+2rh_a+r^2}}\ \textbf{(and analogous)(23)}$$

$B=p^2+\frac{(p-a)(p-b)(p-c)}{p}-4(p-b)(p-c)$(from(1))

$B=p^2+(p-b)(p-c)\left(\frac{p-a}{p}-4\right)$; $\frac{p-a}{p}-4=-\frac{a+3p}{p}$.

$B=p^2-\frac{a+3p}{p}(p-b)(p-c)$.

$p_a^2=\frac{4p^4}{(2p+a)^2}-\frac{4p(a+3p)}{(2p+a)^2}(p-b)(p-c)$.

If $t=p(p-a)\to p_a^2-t=\frac{4p^4}{(2p+a)^2}-\frac{4p(a+3p)}{(2p+a)^2}(p-b)(p-c)-p(p-a)$

$p_a^2-t=\frac{4p^4-4p(a+3p)(p-b)(p-c)-p(p-a)(2p+a)^2}{(2p+a)^2}$

$N=4p^4-4p(a+3p)(p-b)(p-c)-p(p-a)(2p+a)^2$.

By complete expansion and regrouping (direct algebraic calculations), the factor is obtained:

$$N=p(a+3p)(a^2-4bc+4bp+4cp-4p^2)$$

$p_a^2-t=\frac{p(a+3p)(a^2-4bc+4bp+4cp-4p^2)}{(a+2p)^2}$

Substitute $p=\frac{a+b+c}{2}$, $4bp=2b(a+b+c)$, $4cp=2c(a+b+c)$,

$$4p^2 = (a+b+c)^2$$

$$a^2 - 4bc + 4bp + 4cp - 4p^2 = a^2 - 4bc + 2b(a+b+c) + 2c(a+b+c) - (a+b+c)^2$$

$$a^2 - 4bc + 4bp + 4cp - 4p^2 = b^2 + c^2 - 2bc = (b-c)^2$$

$$\mathbf{p_a^2 - t = \frac{p(a+3p)(b-c)^2}{(a+2p)^2} \rightarrow p_a^2 = p(p-a) + \frac{p(a+3p)}{(a+2p)^2}(b-c)^2 \text{ (and analogous)(24)}}$$

REFERENCES

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