ROMANIAN MATHEMATICAL MAGAZINE

36 METRIC RELATIONSHIPS IN A NEW GEOMETRICAL CONFIGURATION

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In this article we worked in a new geometric configuration inspired by Dan Sitaru's work "METRIC RELATIONSHIPS IN ŞAHIN'S TRIANGLE"-www.ssmrmh.ro. We define a similar triangle ourselves and found metric relationships in this triangle.

THEOREM

Let ΔABC be an acute triangle and $X \in Int(\Delta ABC)$ such that $XR \perp BC$; $XQ \perp AC$;

$XP \perp AB$; $|XR| = h_a$; $|XQ| = h_b$; $|XP| = h_c$ (such in figure).

Notations:

$$F = area\ of\ the\ original\ \Delta ABC$$

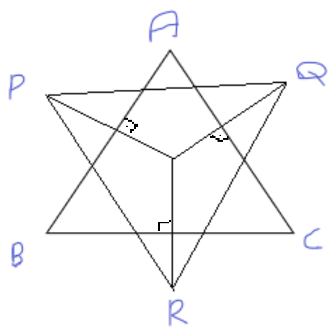
$$m_a, m_b, m_c = medians\ in\ the\ original\ \Delta ABC$$

$$\alpha, \beta, \vartheta = angles\ of\ the\ original\ \Delta ABC$$

$$a, b, c = sides\ of\ original\ \Delta ABC$$

$$a', b', c' = sides\ of\ \Delta PQR$$

$$R^*, R = \ circumradii\ of\ \Delta PQR\ and\ \Delta ABC$$



In these conditions:

$$1.\ a' = \frac{a.m_a}{R}, b' = \frac{b.m_b}{R}, c' = \frac{c.m_c}{R}$$

Proof: $(a')^2 = \frac{4F^2}{b^2} + \frac{4F^2}{b^2} + \frac{4F^2 cos\alpha}{2bc} = \frac{8F^2c^2+8F^2b^2+8F^2(b^2+c^2-a^2)}{2b^2c^2} = \frac{4F^2(2b^2+2c^2-a^2)}{b^2c^2} = \frac{F^2(m_a)^2}{b^2c^2} = \frac{a^2(m_a)^2}{R^2}$

2. $\boldsymbol{Area(\Delta XPQ) = F.(\alpha\ )}$

Proof: $\frac{1}{2}h_c h_b \sin\sin 180 - \alpha\ = \frac{1}{2}h_c h_b \sin\sin \alpha\ = \frac{1}{2}h_c h_b \frac{h_b}{c} = \frac{4F^3}{b^2c^2} = F(\alpha\ )$

3. $\boldsymbol{Area(\Delta XQR) = F.(\beta\ )}$

Proof: It's found similary to 2.

4. $\boldsymbol{Area(\Delta XRP) = F.(\vartheta\ )}$

Proof: It's found similary to 2.

5. $\boldsymbol{Area(\Delta PQR) = F.(\alpha\ + \beta\ + \vartheta\ )}$

Proof: Equations 2,3,4 are found by adding them side by side.

6. $\boldsymbol{\frac{Ra'b'c'}{R^*abc} = \alpha\ + \beta\ + \vartheta\ )}$

Proof: $Area(\Delta PQR) = \frac{a'b'c'}{4R^*} = F.(\alpha\ + \beta\ + \vartheta\ ) = \frac{abc}{4R}(\alpha\ + \beta\ + \vartheta\ )$

$\Rightarrow \frac{Ra'b'c'}{R^*abc} = \alpha\ + \beta\ + \vartheta\ )$

7. $\boldsymbol{\frac{m_a m_b m_c}{R^* R^2} = \alpha\ + \beta\ + \vartheta\ )}$

Proof: In the equation, replace a', b', c' with the equations 1 and the equality is found.

$$\frac{Ra'b'c'}{R^*abc} = \alpha\ + \beta\ + \vartheta\ ) \Rightarrow \frac{Rabcm_a m_b m_c}{R^*abcR^3} = \alpha\ + \beta\ + \vartheta\ ) \Rightarrow$$

$$\Rightarrow \frac{m_a m_b m_c}{R^* R^2} = \alpha\ + \beta\ + \vartheta\ )$$

8. $\boldsymbol{Area(\Delta PQR) = \frac{4F^3.(a^2+b^2+c^2)}{a^2b^2c^2}}$

Proof: It is used that $\alpha\ + \beta\ + \vartheta\ ) = \frac{(a^2+b^2+c^2)}{4R^2}$ and $R = \frac{abc}{4F}$ and equality is found.

9. $\boldsymbol{Area(\Delta PQR) = \frac{F.(m_a^2+m_b^2+m_c^2)}{3R^2}}$

Proof: It is used that $3(a^2 + b^2 + c^2) = 4(m_a^2 + m_b^2 + m_c^2)$ and equality is found.

10. $Area(\Delta PQR) = \frac{16F^3.(m_a^2+m_b^2+m_c^2)}{3a^2b^2c^2}$

Proof: It is found by writing $R = \frac{abc}{4F}$ in equation 9.

11. $Area(\Delta PQR) = (3 - \alpha - \beta - \vartheta)F$

Proof: It is found by writing $\alpha + \beta + \vartheta) = (3 - \alpha - \beta - \vartheta)$ in equation 5.

12. $Area(AQBRCP) = 3F$

Proof: $Area(AQBRCP) = Area(APBX) + Area(BRCX) + Area(CQAX) = \frac{ah_a+bh_b+ch_c}{2} = 3F$

13. $Area(\Delta PQR) = 2.R^2 sin\alpha sin\beta sin\vartheta(\alpha + \beta + \vartheta)$

Proof: It is used that $F = 2R^2 sin\alpha sin\beta sin\vartheta$ and equality is found.

14. $Area(\Delta PQR) = (2 + cos\ cos\ \alpha\ \ cos\ cos\ \beta\ \ cos\ cos\ \vartheta)F$

Proof: It is used that $\alpha + \beta + \vartheta = 180 \Rightarrow \alpha + \beta + \vartheta = 2 + cos\ cos\ \alpha\ \ cos\ cos\ \beta$ $cos\ cos\ \vartheta$ and equality is found.

15. $Area(\Delta PQR) \geq \frac{\sqrt{3}F^2}{R^2}$

Proof: $Area(\Delta PQR) = F.(\alpha + \beta + \vartheta) = F.\frac{(a^2+b^2+c^2)}{4R^2}$

It used to $a^2 + b^2 + c^2 \geq 4\sqrt{3}F$ and equality is found.

16. $|XK|^2 + |XL|^2 + |XM|^2 \geq \frac{4F^2}{9R^2}$

17. $(|AX| + |BX| + |CX|)^2 \geq \frac{16F^2}{9R^2} + 8(|XM||XL| + |XM||XK| + |XK||XL|)$

18. $Area(\Delta PQR) = \frac{Fm_am_bm_c}{R^*R^2}$

19. $\frac{aa'}{R_1} + \frac{bb'}{R_2} + \frac{cc'}{R_3} = \frac{m_am_bm_c}{R^*}$

20. $sin\ sin\ (RPQ) = \frac{R^2 sinsin\ \vartheta\ (\alpha + \beta + \vartheta)}{m_am_b}$

21. $(RQP) = \frac{R^2 sinsin\ \beta\ (\alpha + \beta + \vartheta)}{m_am_c}$

22. $sin\ sin\ (PRQ) = \frac{R^2 sinsin\ \alpha\ (\alpha + \beta + \vartheta)}{m_bm_c}$

23. $\sin \sin (RPQ) = \frac{\sin\sin \vartheta\,(a^2+b^2+c^2)}{4m_a m_b}$

24. $\sin \sin (RQP) = \frac{\sin\sin \beta\,(a^2+b^2+c^2)}{4m_a m_c}$

25. $\sin \sin (PRQ) = \frac{\sin\sin \alpha\,(a^2+b^2+c^2)}{4m_b m_c}$

26. $(a' + b' + c')^2 \leq \frac{3}{4}(a^2 + b^2 + c^2)^2$

27. $(a' + b' + c')^2 \leq \frac{27(a^2+b^2+c^2)}{4}$

28. $(a' + b' + c')^2 \leq \frac{243R^2}{4}$

29. $\frac{\sin\sin RQP}{\sin\sin RPQ} = \frac{\sin\beta . m_b}{\sin\vartheta . m_c}$

30. $\frac{\sin\sin RQP}{\sin\sin PRQ} = \frac{\sin\beta . m_b}{\sin\alpha . m_a}$

31. $\frac{\sin\sin RPQ}{\sin\sin PRQ} = \frac{\sin\vartheta . m_c}{\sin\alpha . m_a}$

32. $\frac{\sin\sin RPQ}{\sin\vartheta . m_c} = \frac{\sin\sin PRQ}{\sin\alpha . m_a} = \frac{\sin\sin RQP}{\sin\beta . m_b} = \frac{1}{R^*} = \frac{a^2+b^2+c^2}{4m_a m_b m_c}$

33. $\frac{\sin\sin RPQ}{\sin\vartheta . m_c} = \frac{\sin\sin PRQ}{\sin\alpha . m_a} = \frac{\sin\sin RQP}{\sin\beta . m_b} \leq \frac{9R^2}{4m_a m_b m_c}$

34. $\frac{\sin\sin RPQ}{\sin\vartheta . m_c} = \frac{\sin\sin PRQ}{\sin\alpha . m_a} = \frac{\sin\sin RQP}{\sin\beta . m_b} \geq \frac{\sqrt{3}F}{m_a m_b m_c}$

35. $R^* = \frac{4m_a m_b m_c}{a^2+b^2+c^2}$

36. $R^* \leq \frac{3}{2} . R$

Reference:

Daniel Sitaru – "Metric relationships in Şahin's triangle" – www.ssmrmh.ro

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