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In any ΔABC the following relationship holds :

$$\sum_{cyc}\frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}}\left(\cot^4\frac{B}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)+\cot^4\frac{C}{2}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)\right)}{\cot^2\frac{B}{2}.\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}}+\cot^2\frac{C}{2}.\sqrt{\cot\frac{C}{2}\cot\frac{A}{2}}}\geq 108$$

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$$\forall A',B',C',x',y',z'>0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\underbrace{\overset{\text{Walter Janous}}{\geq}}_{①}\sqrt{3\sum_{cyc}A'B'}$$

$$\text{Now, }\sum_{cyc}\frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}}\left(\cot^4\frac{B}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)+\cot^4\frac{C}{2}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)\right)}{\cot^2\frac{B}{2}.\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}}+\cot^2\frac{C}{2}.\sqrt{\cot\frac{C}{2}\cot\frac{A}{2}}}$$

$$=\sum_{cyc}\frac{\frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}}}{\cot^2\frac{A}{2}}\left(\frac{\cot^2\frac{C}{2}\cot^2\frac{A}{2}}{\cot^2\frac{B}{2}}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)^2+\frac{\cot^2\frac{A}{2}\cot^2\frac{B}{2}}{\cot^2\frac{C}{2}}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)^2\right)}{\frac{\sqrt{\cot\frac{C}{2}\cot\frac{A}{2}}}{\cot^2\frac{B}{2}}+\frac{\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}}}{\cot^2\frac{C}{2}}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x'=\frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}}}{\cot^2\frac{A}{2}},y'=\frac{\sqrt{\cot\frac{C}{2}\cot\frac{A}{2}}}{\cot^2\frac{B}{2}},z'=\frac{\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}}}{\cot^2\frac{C}{2}},A'=\frac{\cot^2\frac{B}{2}\cot^2\frac{C}{2}}{\cot^2\frac{A}{2}}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)^2,\right.$$
$$\left.B'=\frac{\cot^2\frac{C}{2}\cot^2\frac{A}{2}}{\cot^2\frac{B}{2}}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)^2,C'=\frac{\cot^2\frac{A}{2}\cot^2\frac{B}{2}}{\cot^2\frac{C}{2}}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)^2\right)$$

$$\overset{\text{via }①}{\geq}\sqrt{3\sum_{cyc}\left(\frac{\cot^2\frac{B}{2}\cot^2\frac{C}{2}}{\cot^2\frac{A}{2}}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)^2.\frac{\cot^2\frac{C}{2}\cot^2\frac{A}{2}}{\cot^2\frac{B}{2}}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)^2\right)}$$

$$\overset{AM-GM}{\geq}\sqrt{3\sum_{cyc}\cot^4\frac{C}{2}.\left(\left(4\cot\frac{B}{2}\cot\frac{C}{2}\right).\left(4\cot\frac{C}{2}\cot\frac{A}{2}\right)\right)}\overset{AM-GM}{\geq}3.\sqrt[3]{4^6\left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^8}$$

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$$= 12.\left(\frac{s}{r}\right)^{\frac{4}{3}} \overset{\text{Mitrinovic}}{\geq} 12.\left(3\sqrt{3}\right)^{\frac{4}{3}} = 108 \text{ and so,}$$

$$\sum_{cyc} \frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}}\left(\cot^4\frac{B}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)+\cot^4\frac{C}{2}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)\right)}{\cot^2\frac{B}{2}.\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}}+\cot^2\frac{C}{2}.\sqrt{\cot\frac{C}{2}\cot\frac{A}{2}}} \geq 108 \ \forall \Delta ABC,$$

$''='' \text{ iff } \Delta ABC \text{ is equi}\textit{la}\text{ter}\textit{al}\ (QED)$