ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship ho$l$ds :

$$\sum_{cyc}\frac{\sqrt{m_b m_c}.\left(m_b^4(m_a+m_b)^2+m_c^4(m_c+m_a)^2\right)}{m_b^2.\sqrt{m_a m_b}+m_c^2.\sqrt{m_c m_a}}\geq 3\cdot(18r^2)^2$$

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$$\forall\, A', B', C', x', y', z' > 0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\underbrace{\overset{\text{Walter Janous}}{\geq}}_{①}\sqrt{3\sum_{cyc}A'B'}$$

$$\text{Now, }\sum_{cyc}\frac{\sqrt{m_b m_c}.\left(m_b^4(m_a+m_b)^2+m_c^4(m_c+m_a)^2\right)}{m_b^2.\sqrt{m_a m_b}+m_c^2.\sqrt{m_c m_a}}$$

$$=\sum_{cyc}\frac{\frac{\sqrt{m_b m_c}}{m_a^2}.\left(\frac{m_c^2 m_a^2}{m_b^2}(m_c+m_a)^2+\frac{m_a^2 m_b^2}{m_c^2}(m_a+m_b)^2\right)}{\frac{\sqrt{m_c m_a}}{m_b^2}+\frac{\sqrt{m_a m_b}}{m_c^2}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(\begin{array}{c} x'=\frac{\sqrt{m_b m_c}}{m_a^2}, y'=\frac{\sqrt{m_c m_a}}{m_b^2}, z'=\frac{\sqrt{m_a m_b}}{m_c^2}, \\ A'=\frac{m_b^2 m_c^2}{m_a^2}(m_b+m_c)^2, B'=\frac{m_c^2 m_a^2}{m_b^2}(m_c+m_a)^2, C'=\frac{m_a^2 m_b^2}{m_c^2}(m_a+m_b)^2\end{array}\right)\overset{\text{via } ①}{\geq}$$

$$\sqrt{3\sum_{cyc}\left(\frac{m_b^2 m_c^2}{m_a^2}(m_b+m_c)^2.\frac{m_c^2 m_a^2}{m_b^2}(m_c+m_a)^2\right)}\overset{\text{AM-GM}}{\geq}\sqrt{3\sum_{cyc}m_c^4.\left((4m_b m_c).(4m_c m_a)\right)}$$

$$\overset{\text{AM-GM}}{\geq} 3.\sqrt{\sqrt[3]{4^6(m_a m_b m_c)^8}}\overset{\text{Lascu + AM-GM}}{\geq} 12.\left(\sqrt{s(s-a)}.\sqrt{s(s-b)}.\sqrt{s(s-c)}\right)^{\frac{4}{3}}$$

$$=12.(rs^2)^{\frac{4}{3}}\overset{\text{Mitrinovic}}{\geq}12.(27r^3)^{\frac{4}{3}}=12(3r)^4=3.(18r^2)^2 \text{ and so,}$$

$$\sum_{cyc}\frac{\sqrt{m_b m_c}.\left(m_b^4(m_a+m_b)^2+m_c^4(m_c+m_a)^2\right)}{m_b^2.\sqrt{m_a m_b}+m_c^2.\sqrt{m_c m_a}}\geq 3.(18r^2)^2\ \forall\,\Delta\,\text{ABC},$$

$$''='' \text{ iff } \Delta\text{ ABC is equilateral (QED)}$$