ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a r_b\left(r_b^3+r_c r_a^2\right)}{r_c^3\left(r_a^3+r_b^3\right)}+\frac{r_c r_b\left(r_c^3+r_a r_b^2\right)}{r_a^3\left(r_c^3+r_b^3\right)}+\frac{r_a r_c\left(r_a^3+r_b r_c^2\right)}{r_b^3\left(r_a^3+r_c^3\right)}\geq\frac{2}{R}$$

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Let us formalite the left side of the given expression according to the conditions of Walter Janous' theorem:

$$LHS=\frac{(r_c r_b)^3}{(r_a r_b)^3+(r_a r_c)^3}\left(\frac{r_c}{r_b^2}+\frac{r_a}{r_c^2}\right)+$$

$$\frac{(r_c r_a)^3}{(r_c r_b)^3+(r_a r_b)^3}\left(\frac{r_a}{r_c^2}+\frac{r_b}{r_a^2}\right)+\frac{(r_a r_b)^3}{(r_a r_c)^3+(r_b r_c)^3}\left(\frac{r_b}{r_a^2}+\frac{r_c}{r_b^2}\right)$$

$$If\ \ x,y,z,a',b',c'\in R^+\ and$$

$$\frac{x}{y+z}(b'+c')+\frac{y}{z+x}(c'+a')+\frac{z}{x+y}(a'+b')\geq\sqrt{3(a'b'+b'c'+c'a')}$$

By Walter Janous' inequality, we have:

$$(r_c r_b)^3=x,\qquad (r_c r_a)^3=y,\qquad (r_a r_b)^3=z\quad and$$

$$\frac{r_b}{r_a^2}=a',\qquad \frac{r_c}{r_b^2}=b',\qquad \frac{r_a}{r_c^2}=c'$$

$$LHS=\sqrt{3\left(\frac{r_b}{r_a^2}\cdot\frac{r_c}{r_b^2}+\frac{r_c}{r_b^2}\cdot\frac{r_a}{r_c^2}+\frac{r_a}{r_c^2}\cdot\frac{r_b}{r_a^2}\right)}\overset{A-G}{\geq}$$

$$\sqrt{3.3\sqrt[3]{\frac{(r_a.r_b.r_c)^2}{(r_a.r_b.r_c)^4}}}=3\sqrt[6]{\frac{1}{(r_a.r_b.r_c)^2}}=3\sqrt[3]{\frac{1}{r_a.r_b.r_c}}\overset{(*)}{\geq}3\sqrt[3]{\frac{8}{27R^3}}=\frac{2}{R}$$

$$r_a.r_b.r_c=\frac{F^2}{r}\ \to\ \frac{1}{r_a.r_b.r_c}=\frac{1}{rp^2}\geq\frac{2}{R}\cdot\frac{4}{27R^2}=\frac{8}{27R^3}\ (*)$$

$$Equality\ \ holds\ \ for:a=b=c.$$