ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{w_a^2}{r_b+r_c}+\frac{w_b^2}{r_c+r_a}+\frac{w_c^2}{r_a+r_b}\geq\frac{9r^2}{R}$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$Lemma\ 1$: $w_a=\frac{2bc}{b+c}\cdot cos\frac{A}{2}$

$Lemma\ 2$: $r_b+r_c=4R\cdot\cos^2\frac{A}{2}$

$Proof$: $r_b+r_c=s\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=$

$$=\frac{s\cdot\sin\left(\frac{B+C}{2}\right)\cdot\cos\frac{A}{2}}{\prod\cos\frac{A}{2}}=\frac{s\cdot\cos^2\frac{A}{2}}{\frac{s}{4R}}=4R\cdot\cos^2\frac{A}{2}$$

$Lemma\ 3$: $\sum\frac{1}{a^2}\leq\frac{1}{4r^2}$

$Proof$: $r=\frac{F}{s},\ r_a=\frac{F}{s-a}$

$$rr_a=\frac{F^2}{s(s-a)}=(s-b)(s-c)\leq\frac{(s-b+s-c)^2}{4}=\frac{a^2}{4}\Rightarrow rr_a\leq\frac{a^2}{4}\Rightarrow a^2\geq 4rr_a$$

$$\sum\frac{1}{a^2}\leq\sum\frac{1}{4rr_a}=\frac{1}{4r}\sum\frac{1}{r_a}=\frac{1}{4r^2}$$

$$LHS=\sum\frac{\left(\frac{2bc}{b+c}\right)^2\cdot cos\frac{A}{2}}{4R\cdot\cos^2\frac{A}{2}}=\frac{1}{4R}\cdot\sum\left(\frac{2ac}{a+c}\right)^2=\frac{1}{R}\cdot\sum\left(\frac{ac}{a+c}\right)^2=\frac{1}{R}\cdot\sum\frac{1}{\left(\frac{1}{a}+\frac{1}{c}\right)^2}\geq$$

$$\geq\frac{1}{R}\cdot\frac{9}{\sum\left(\frac{1}{a}+\frac{1}{c}\right)^2}\geq\frac{1}{R}\cdot\frac{9}{2\sum\frac{2}{a^2}}=\frac{1}{R}\cdot\frac{9}{4\sum\frac{1}{a^2}}\geq\frac{1}{R}\cdot\frac{9}{4\cdot\frac{1}{4r^2}}=\frac{9r^2}{R}$$

$Equality\ holds\ for\ a=b=c.$