ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\left(\frac{r_a}{w_b w_c}\right)^2+\left(\frac{r_b}{w_c w_a}\right)^2+\left(\frac{r_c}{w_a w_b}\right)^2 \geq \left(\frac{3}{s}\right)^2$$

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$$Lemma\ 1:\ \frac{r_a r_b r_c}{w_a w_b w_c}=\frac{(a+b)(b+c)(a+c)}{8abc}\geq\frac{8abc}{8abc}=1$$

$$\mathrm{Pr}\,oof:\ r_a=\frac{F}{s-a},\ r_b=\frac{F}{s-b},\ r_c=\frac{F}{s-c}$$

$$w_a=\frac{2}{b+c}\sqrt{bcs(s-a)},\ w_b=\frac{2}{a+c}\sqrt{acs(s-b)},\ w_c=\frac{2}{a+b}\sqrt{abs(s-c)}$$

$$\prod w_a=\frac{8sabc}{(a+b)(b+c)(c+a)}\cdot\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\frac{8abc}{(a+b)(b+c)(c+a)}\cdot Fs\ \dots(1).$$

$$\prod r_a=\frac{F^3}{(s-a)(s-b)(s-c)}=Fs\ \dots(2).$$

$$(1)\ \wedge\ (2)\ \Rightarrow\ \prod w_a=\frac{8abc}{(a+b)(b+c)(c+a)}\cdot\prod r_a\ \Rightarrow\ \frac{\prod r_a}{\prod w_a}=\frac{(a+b)(b+c)(c+a)}{8abc}$$

$$Lemma\ 2:\ w_a^2\leq s(s-a)$$

$$\mathrm{Pr}\,oof:\ w_a^2=\frac{4bc}{(b+c)^2}\cdot s(s-a)\leq\frac{4bc}{4bc}\cdot s(s-a)=s(s-a)$$

$$LHS=\sum\left(\frac{r_a}{w_b w_c}\right)^2\geq 3\cdot\sqrt[3]{\left(\frac{r_a r_b r_c}{(w_a w_b w_c)^2}\right)^2}=3\sqrt[3]{\left(\frac{r_a r_b r_c}{w_a w_b w_c}\right)^2\cdot\frac{1}{(w_a w_b w_c)}}\geq$$

$$\geq 3\sqrt[3]{\frac{1}{s^3(s-a)(s-b)(s-c)}}=\frac{3}{s}\sqrt[3]{\frac{27}{27(s-a)(s-b)(s-c)}}\geq\frac{9}{s}\sqrt[3]{\frac{1}{(3s-a-b-c)^3}}=\frac{9}{s^2}$$

$$Equality\ holds\ for\ a=b=c.$$