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In ΔABC the following relationship holds:

$$\sum_{cyc} a^2 cos\frac{B}{2}cos\frac{C}{2} \ge 27r^2$$

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$$\sum_{cyc} a^2 cos\frac{B}{2}cos\frac{C}{2} = \sum_{cyc} a^2\sqrt{\frac{s(s-b)}{ac}}\cdot\sqrt{\frac{s(s-c)}{ab}} =$$

$$= \sum_{cyc} as\sqrt{\frac{(s-b)(s-c)}{bc}} = s\sum_{cyc} asin\frac{A}{2} \overset{CEBYSHEV}{\ge}$$

$$\ge \frac{s}{3}\sum_{cyc} a\sum_{cyc} sin\frac{A}{2} \overset{JENSEN}{\ge} \frac{s}{3}\cdot 2s\cdot 3sin\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right) =$$

$$= 2s^2 sin\frac{A+B+C}{6} = 2s^2 sin\frac{\pi}{6} \overset{MITRINOVICI}{\ge} 2\cdot\left(3\sqrt{3}r\right)^2\cdot\frac{1}{2} = 27r^2$$

Equality holds for an equilateral triangle.