ROMANIAN MATHEMATICAL MAGAZINE

If $n \geq 3$ then in ΔABC the following relationship holds:

$$\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}+\frac{R^n}{r^{n-1}} \geq 2^n r+\frac{\sqrt[3]{abc}}{3}$$

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$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \overset{CBS}{\leq} \sqrt{3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)} \overset{Steining}{\leq} \sqrt{\frac{3}{4r^2}}=\frac{\sqrt{3}}{2r} \ (1)$$

$$abc=4Rrs \overset{Euler\ \&\ Mitrinovic}{\leq} 4.R.\frac{R}{2}.\frac{3\sqrt{3}R}{2}=\left(\sqrt{3}R\right)^3 \ (2)$$

$$We\ need\ to\ show:$$

$$\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}+\frac{R^n}{r^{n-1}} \geq 2^n r+\frac{\sqrt[3]{abc}}{3}$$

$$\frac{1}{\frac{\sqrt{3}}{2r}}+\frac{R^n}{r^{n-1}} \geq 2^n r+\frac{\sqrt{3}R}{3} \ or \ \frac{2r}{\sqrt{3}}+\frac{R^n}{r^{n-1}} \geq 2^n r+\frac{\sqrt{3}R}{3}$$

$$\frac{2}{\sqrt{3}}+\left(\frac{R}{r}\right)^n \geq 2^n+\frac{R}{r\sqrt{3}} \ or \ \sqrt{3}(x^n-2^n)-(x-2) \overset{\frac{R}{r}=x\geq 2\ Euler}{\geq} 0$$

$$\sqrt{3}(x-2)(x^{n-1}+2x^{n-2}+4.x^{n-3}\ldots+2^{n-1})-(x-2) \geq 0$$

$$(x-2)\left(\sqrt{3}(x^{n-1}+2x^{n-2}+4.x^{n-3}\ldots+2^{n-1})-1\right) \geq 0$$

$$true\ \ as\ x \geq 2\ and\ for\ n \geq 3, (x^{n-1}+2x^{n-2}+4.x^{n-3}\ldots+2^{n-1})-1>0$$

Equality holds for an equilateral triangle.