ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{bc}{b^2+c^2} \le \frac{R^2}{4r^2}\sum \frac{a^2}{b^2+c^2} \le \frac{3R^4}{32r^4}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\sum \frac{a^2}{b^2+c^2} \overset{AM-GM}{\le} \sum \frac{a^2}{2bc} = \frac{1}{2abc}\sum a^3 = \frac{2s(s^2-3r^2-6Rr)}{8Rrs} \overset{Gerretsen}{\le}$$

$$\le \frac{4R^2-2Rr}{4Rr} = \frac{R}{r}-\frac{1}{2}$$

$$then\ \frac{R^2}{4r^2}\sum \frac{a^2}{b^2+c^2} \le \frac{R^2}{4r^2}\left(\frac{R}{r}-\frac{1}{2}\right) = \frac{R^3}{4r^3}-\frac{R^2}{8r^2}$$

$$We\ need\ to\ show:$$

$$\frac{R^2}{4r^2}\sum \frac{a^2}{b^2+c^2} \le \frac{3R^4}{32r^4}\quad or$$

$$\frac{R^3}{4r^3}-\frac{R^2}{8r^2} \le \frac{3R^4}{32r^4}\ or\ 3x^2-8x+4 \overset{\frac{R}{r}=x\ge 2}{\ge} 0\ or (x-2)(3x-2)\ge 0\ \ true\ (A)$$

$$\sum \frac{bc}{b^2+c^2} \overset{AM-GM}{\le} \frac{1}{2}\sum \frac{bc}{bc} = \frac{3}{2}\ (B)$$

$$\frac{R^2}{4r^2}\sum \frac{a^2}{b^2+c^2} \overset{Nesbitt}{\ge} \frac{3}{2}\frac{R^2}{4r^2} \overset{Euler}{\ge} \frac{3}{2}\ (C)$$

$$From\ (A),(B),(C) we\ have: \sum \frac{bc}{b^2+c^2} \le \frac{R^2}{4r^2}\sum \frac{a^2}{b^2+c^2} \le \frac{3R^4}{32r^4}$$

Equality holds for an equilateral triangle.